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UDK 513 **MATHEMATICS**

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On the Area of a Region on a Developable Surface

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1. A family of tangent lines to a space curve Γ forms a *developable surface* (DS). The position of a point A on DS is defined by distance $u = AM$ along the tangent to Γ at the point M, and by the arc length v locating the point of tangency M on Γ. The coefficients of the first differential Gaussian form for DS are known [1]: $E=1$, $F=1$, $G=1+(u/\rho)^2$, where $1/\rho$ is the curvature of Γ. We introduce new coordinates (u, φ), where φ is the angle swept on DS by the tangent line to Γ from its initial position $\varphi = 0$. Then the area of a region Ω on DS, bounded by two curves $u(\varphi)$ and Γ, $u(\varphi) = 0$, and two tangent lines with angles φ_1 and φ_2, is given by the expression

$$S = \iint_{\Omega} \sqrt{EG - F^2}\, du dv = \iint_{\Omega} u\, du d v/\rho = \frac{1}{2}\int_{v_1}^{v_2} u^2 d v/\rho$$

and, by definition of curvature $1/\rho = d\varphi/dv$, equals:

$$S = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} u^2(\varphi) d\varphi \tag{1}$$

We observe that *S does not explicitly depend on length v of arc Γ*.

Construct the *conical hodograph* of the region Ω by translating (parallel to itself) each tangent segment $u(\varphi)$ so that all their starting points join one common point M_0, the *vertex* of the hodograph. By choosing the point M_0 as the curve Γ, we notice that, on the other hand, expression (1) defines polar area on a conical surface. Therefore, we have the following

T h e o r e m. *The area of any region on a developable surface equals the area of its hodograph* (*on a conical surface*).

Another, simple proof of the theorem follows from the area equality of an elementary triangle on DS, between two infinitesimally close tangents to Γ and the curve u (φ); and of the corresponding triangle on the hodograph. Here is an even simpler proof: a curve Γ, by definition, is a limiting case of a circumscribed polygon. As for any rectilinear curve, it is obvious that the hodograph theorem holds *exactly*.

We will say that a line *segment rotates around a curve* Γ if it always lies on a tangent line to that curve. During such rotation the segment sweeps an area which does not depend on the size of the arc of Γ, but only on the rule of changing of the endpoints of the segment in dependence on its direction in space. If we change the size of Γ keeping it *similar to itself*, then the swept area will remain *unchanged*. In the limit, the shrinking of the arc Γ to a point results in an ordinary rotation of the segment on the hodograph.

The claim of the theorem allows a vivid dynamic interpretation. During an infinitesimally small rotation around the arc Γ, the segment makes two instantaneous motions; rotation around the point of tangency (accompanied by stretching u (φ)), and sliding along itself. As the segment slides along itself it sweeps no area, the entire area is swept during rotations around the points of tangency, they do correspond to the rotation on the hodograph. The case of rotation of a segment around a polygon is trivial: the areal regions swept around the vertices, if pushed together, "add up" to the hodograph.

We'll illustrate the hodograph theorem on a series of examples of finding areas. Then we extend the theorem in regard to finding volumes of "rotations" around cylinders and cones as well as special solids of revolutions. Note that the above two simple proofs of the theorem and its vivid dynamic interpretation allow us to regard both the theorem and the solutions below in sections 2 - 5 (and some others) as *elementary*.

2. We consider examples for a particular case when Γ is a *plane* curve. Actually, our goal will be *constructing the hodographs*. They are built using elementary geometric arguments, and represent simple shapes. The area of the hodograph, according to the theorem, gives us the area of the *sought* initial complex shape. Examples marked by asterisks present new results, others are known from calculus (see, e.g., [2]).

E x a m p l e s. a)* A line segment of constant length a rotating around a simple closed curve sweeps an "*oval" ring*. Hodograph of such a ring is a *circle* of radius a with the same area πa^2, regardless on the size and the shape of the inner curve. In the known example of a *circular* ring, this area is found by using the Pythagorean theorem.

b) *Tractrix* is a curve with constant length a of the tangent segment from the curve to its asymptote. The hodograph of (two-sided) tractrix is a *semicircular disk* of radius a, therefore its area is $\pi a^2/2$.

The hodograph of *symmetrically cut tractrix* forms a *segment of the circle* cut off at distance from its center equal to the width of the cut.

c)* *Generalized tractrix*. The region between the paths traced by the bicycle wheels is swept by rotating segment a (between the two axis of the wheels) around the rear wheel path. Its hodograph is a *circular sector* with radius a. Examples a) and b) follow from here as special cases.

When a bicycle turns from one direction to a perpendicular one, this area equals $\pi a^2/4$, *regardless* of how it turns.

d) *Cycloid* is a trajectory of a point on a circle rolling on a line. At any instant the circle rotates around its bottom point. So the tangent to the cycloid is always directed towards the upper point of the rolling circle, thus being a chord of that circle. This tangent segment, rotating around the cycloid, sweeps the arch-like region of the circumscribing rectangle above the cycloid. Thus the hodograph of that *arch* is the rolling *disk*.

Symmetrically cut section of this arch has as hodograph a *segment* of the rolling disk, with height equal to the width of the cut.

e)* *Pursuit curve* is a trajectory of a dog chasing and always aiming at running fox. If their speeds are equal, and the fox runs along a straight line, then during the entire pursuit the sum of the segment "dog-fox" and its projection on the fox's path remains constant. Therefore, the hodograph of the region between the two paths, and the initial "dog-fox" line, is a *parabolic* sector, whose focus at the vertex of the hodograph.

3. Let the plane curve Γ be the profile of a cylinder; and let a region Φ be in the plane tangent to the cylinder. Let this plane *rotate around the cylinder* remaining always tangent to its surface. Then the region Φ sweeps some solid of "rotation". In each section of this spacial picture parallel to the plane of Γ, the hodograph theorem holds, therefore the following is valid:

Corollary 1. *The volume of the solid of rotation of a plane region around a cylinder does not depend on the profile of the cylinder, and equals the volume of its hodograph which is formed by the corresponding revolution of the region around a fixed axis.*

E x a m p l e s: a) A *sphere with a cylindrical hole* is swept by the semi-circular disk rotating around the cylinder. Therefore the volume of the (wedding) ring equals that of the *sphere* with diameter equal to the height of the hole. Analogous result holds for non-circular cylinders.

b) The section of a circular cone parallel to its axis is a hyperbola. If this section rotates around the axis of the cone, it envelopes some cylinder. On one hand, the hyperbolic segment sweeps the *cone with a cylindrical hole*, whose volume is easy to find. On the other hand, this hyperbolic segment rotating around its axis forms the hodograph, a segment of a *hyperboloid of two sheets*, with the same volume.

c)* A *hyperboloid of one sheet* is obtained by rotating a plane angle around a cylinder. Its volume is the sum of the volumes of the inscribed cylinder and the hodograph, which is a *cone* obtained by rotating the angle around its edge.

4. By analogy with Section 3, consider the rotation of a plane region around a conical surface. From similarity of curves Γ in the parallel sections of the cone, we conclude:

Corollary 2. *The volume of the solid of rotation of a plane region around a cone does not depend on its distance from the vertex of the cone. For a circular cone with vertex angle* 2α *this volume equals that of the hodograph times* $\cos\alpha$.

E x a m p l e s: a) A *sphere with a conical hole* is swept by rotating a semi-circular disk around a cone. Therefore the volume of such ring equals the volume of the *sphere* whose diameter is the height of the hole times $\cos\alpha$.

b) An ellipse, a section of a cylinder, rotating around the axis of the cylinder, envelopes a cone. On one hand, the ellipse is rotating around that cone, sweeping a *cylinder with a funnel hole*, whose volume is 2/3 of that of the cylinder. On the other hand this is the volume of the *ellipsoid* of revolution (hodograph) times $\cos\alpha$. Using the factor $\cos\alpha$, the volume of an ellipsoid (particularly a *sphere*) is one and half times as large as the volume of its circumscribing cylinder.

c) A parabola is a section of a cone parallel to its generator. When rotating around the cone's axis, it envelopes a similar cone, flipped. When rotating around its own axis it forms a *paraboloid* which is the hodograph of the initial *cone with a conical funnel*, whose volume is easily found. Factor $\cos\alpha$ gives volume of a paraboloid is half that of its circumscribing cylinder.

Note that in the case of rotating around a circular cylinder or cone, volumes can be found without the hodograph theorem, using the property of a circular ring (example a, Section 1), or the Pythagorean theorem (author [3]).

5. The centroid of a triangle is at one-third of the height from the base, therefore rotating around the base the triangle sweeps twice less volume than rotating around an axis parallel to the base and passing through the opposite vertex of the triangle. From here follows

Corollary 3. *Let the end-point of a line segment, rotating around a curve* Γ*, move along a straight line* X*. Then the volume of the solid of revolution of region* Φ*, bounded by the lines* Γ *and* X*, around* X*-axis equals half that of the solid of revolution of the hodograph of the region around the axis* H *passing through the vertex parallel to* X.

E x a m p l e s. a) *Pseudosphere* is obtained by rotating tractrix around its asymptote, the axis X. Because the hodograph of tractrix is a semicircle of radius a, the volume of the pseudosphere equals that of the *sphere* of radius a.

In general, the volume of symmetrically *cut pseudosphere* equals that of a hemisphere with a cylindrical hole of height being the length a_x of the sub-tangent at the cutting point. Or, according to the example a) of Section 3, it equals the volume of a *sphere* of radius a_x.

b)* The *pursuit curve* (example e, Section 2) rotating about the fox's line bounds a solid whose volume equals half that of the *parabolic segment* (vertex at focus of the paraboloid).

6. Consider arbitrary curve Γ. Its natural equation can be easily reduced [2] to the form $u = u(\varphi)$ giving the relation between the arc length and the angle of the tangent to Γ, or reversely $\varphi = \varphi(u)$. Then the hodograph theorem in its *generalized* form allows us to solve also the following

E x a m p l e s:

a)* The area of the region between an *arbitrary curve* $u(\varphi)$ and its *involute* is given by the formula (1). In particular the hodograph of the *involute of a circle* is an *Archimedean spiral*.

b)* The area of one *arch* traced by a point on a circle rolling without slipping along *arbitrary* curve $\varphi(u)$ is K times as large as the area of the rolling circle, where

$$K = 3 - \frac{1}{\pi}\int_0^{2\pi} \varphi(u)\sin u\,du$$

Particular cases: $\varphi(u) = 0$ - cycloid; $\varphi(u) = \pm\, ru/R$ - epi- or hypo- cycloids, $K = 3 \pm 2r/R$.

Similarly, using the hodograph theorem we can also find areas of hyperbola, exponential, pedal of a circle (in particular cardioid), pedal of involute of a circle; volumes of revolution of: exponential around its asymptotes, regions bounded by curves of Perseus (in particular, Cassini ovals, or lemniscates of Booth and Bernoulli), around their axes of symmetry, cycloid around tangent to its upper point.

We have considered only the examples with plane curves Γ. Their analogies obviously hold on developable surfaces, because a developable surface can be unfolded onto a plane, so the formulations of the hodograph theorem in both cases are *equivalent*.

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Abstract (in Armenian)

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On the area of a region on a developable surface

A theorem is proved that the area of any region on a developable surface equals the area of the hodograph of this region on a conical surface.

In applications, the areas and volumes of numerous geometric figures are determined. The interpretation of the theorem and the solution of the examples considered can be classified as elementary.

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