The Generalized Problem of Counterfeit Coins

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Many readers are quite familiar with the following classical problem about counterfeit coins, which is striking in that it is solvable.

The problem of the bag of false coins

There are N bags, each containing sufficiently many coins. All but one bag contain identical "normal" coins, but in one bag all the coins are counterfeit. The weight of a normal coin is known, and it is known that a false coin is one gram lighter than a normal coin. It is required to find the bag with false coins by a single weighing *on scales with a set of weights.*

Here is how this problem can be solved. The bags are numbered consecutively, and from each bag a number of coins equal to the number on the bag is taken. The total weight of all the coins selected in this way will "fall short of" the weight of the same number of normal coins (which we know) by a number of grams equal to the number on precisely that bag containing the false coins.[1]

While reflecting on this problem, I came to the more surprising conclusion that an even more complicated problem can be solved by a single weighing.

The problem of several bags with false coins

Suppose that under the conditions of the preceding problem there are not one but several bags of false coins, and it is not known how many. It is required to find all these bags by a single weighing *on scales with a set of weights.*

After solving this problem, I grew bolder and considered further complications. The problem turned out to be solvable under even more surprising conditions.

The problem of bags with heavy and light coins

Among the N bags are some (unknown) number of bags with heavy coins and some (also unknown) number of bags with light coins, a light coin being one gram lighter than a normal coin and a heavy coin being one gram heavier than a normal coin. It is required to find the bags with normal coins, those with heavy coins, and those with light coins by a single weighing *on scales with a set of weights.* (Recall that all the coins in a particular bag are of the same weight and that the weight of a normal coin is known.)

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[1]This problem can also be solved, albeit with a bit more cleverness, in the case when the weight of a normal coin is unknown and there is no set of weights. Try to figure out how.

The solvability of this problem inspired me to the following almost evident generalization. Up to this point we have actually considered problems of two or three kinds of coins; therefore, it is natural to go further.

The problem of bags with several kinds of coins

Suppose that there are N bags, each containing sufficiently many coins. There are coins of various kinds, but the coins in a particular bag are all of a single kind. The number of bags with coins of a given kind is arbitrary and unknown. Two coins of different kinds differ in weight by an integral number of grams. The set of all possible weights of coins is known. It is required to determine the kind of coins in each bag by means of a single weighing on scales with a set of weights.

Before proceeding to the solution below of the generalized problem of false coins, we propose that the reader try to solve the preceding problems independently.

Solution of the problem of bags with several kinds of coins

We number the bags consecutively from 0 to $N-1$. Denote the weight of the lightest coin by m, and the wieght of the heaviest coin by $m+k$. Suppose that the bag with number j contains coins of weight $m+\Delta_j$, so that Δ_j determines the kind of coins in the jth bag. Depending on the kind of coins, the quantities Δ_j can take (integer) values $0, 1, 2, \ldots, k$.

From the bag with number j we now take k^j coins, that is, 1 from the first bag, k from the second bag, $\ldots$, k^{N-1} from the last bag. The total number of coins taken is

$$M = \sum_{j=0}^{N-1} k^j = 1 + k + k^2 + \cdots + k^{N-1} = \frac{k^N - 1}{k-1}.$$

Their total weight S on the scales is

$$S = \sum_{j=0}^{N-1} (m + \Delta_j) k^j = m \sum_{j=0}^{N-1} k^j + \sum_{j=0}^{N-1} \Delta_j k^j = m \cdot M + \sum_{j=0}^{N-1} \Delta_j k^j.$$

Since Δ_j is always $< k$, the second sum

$$\Delta = \sum_{j=0}^{N-1} \Delta_j k^j = \Delta_0 + \Delta_1 k + \Delta_2 k^2 + \cdots + \Delta_{N-1} k^{N-1}$$

on the right-hand side is the translation of the number Δ from the decimal number system (in which the weights are given) into the number system with base k. In this system Δ is written as a number with the following sequence of digits:

$$(*) \qquad \Delta \to \overline{\Delta_{N-1}\Delta_{N-2}\ldots\Delta_1\Delta_0}.$$

We see that the digits in this expression indicate the kinds of coins in the sequence of bags, taken in reverse order. This is the main idea for our solution.

Accordingly, from the total weight S of all the M coins selected we subtract the weight Mm of the same number of coins of the lightest kind and then represent the difference $\Delta = S - Mm$ in the number system with base k (we decompose into powers of k, beginning with the highest power). This gives us a number of the form $(*)$. Its jth digit from the end (counting from zero) indicates the kind Δ_j of coins in the bag with number j.

Example

In the table below we give the weights of coins contained in five bags. The numbers on the bags are given in the top row of the table, from right to left (this is the reverse order), and the kinds of coins are indicated under the weights of coins in the bags. They are the desired numbers.

TABLE

4	3	2	1	0	number j on bag
11 g	12 g	10 g	12 g	10 g	weight $m + \Delta_j$ of a coin
1	2	0	2	0	kind Δ_j of coin
81	27	9	3	1	number k^j of coins selected

In this case $k = 3$ and the numbers of coins taken correspond to powers of 3, as shown in the last row of the table. We select $M = 121$ coins in all. Their total weight on the scales is $S = 1351$ g. Subtracting $M \cdot m = 121 \cdot 10$, we get $\Delta = 141$ g. Expressing Δ in the ternary number system,

$$\Delta = 1 \cdot 3^4 + 2 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3^1 + 0 \cdot 3^0,$$

we get the number $\overline{12020}$, and its sequence of digits coincides with the original sequence of kinds given in the table.

If $k = 10$, then it is not necessary to translate Δ from one number system into another. For the case $k = 3$ there is an interpretation of the solution of the problem somewhat different from ours. We leave it to the reader to find this interpretation.

A little history

The classical problem of one bag with false coins can be found in many popular books on mathematics. It is said that during the Second World War the English dropped leaflets containing this problem on German troops with the goal of distracting and thus disorganizing them; supposedly, they wasted 40,000 man-hours on solving it.

In the book *Logik macht Spass* by G. Bizám and J. Herczeg (Akad. Kiadó, Budapest, 1976, German translation from the Hungarian) the case of two bags with false coins is considered, and this problem is solved by means of two weighings.

The classical problem of false coins has recently found applications in the theory of coding and information for detecting errors in a code.