THE RELATIVISTIC GENERALIZED THEORY OF GRAVITATION. I

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Astrofizika, Vol. 4, No. 4, pp. 567–579, 1968

The relativistic variant of the generalized theory of gravitation is discussed. The conservation laws are formulated, and a derivation is given of the field equations for the internal region of a spherically symmetric distribution of masses. The external solution, which is used as the condition on the surface of the configuration, is given. Models are developed for massive superdense static configurations with a very high mass defect and a gravitational red shift.

1. A number of new astrophysical phenomena have been discovered during the last decade, but they have not as yet found a theoretical explanation which is free from internal inconsistencies. It is probably premature to suppose that the existing physical theory is incomplete, but there are reasons to doubt that this theory will continue to be valid indefinitely. It follows that, in parallel with studies based on the existing laws of physics, it is not unreasonable to try to explore the solutions of new problems along new directions which may involve modifications of existing laws of physics. The generalized theory of gravitation developed largely by Jordan et al. [1, 2] seems to us to be highly attractive from this point of view, although it has not reached its final stage of development. This problem is also considered in [3]. The theory is based on an idea put forward by Dirac [4] that the gravitational constant may in fact be variable. It is interesting to note that this theory is consistent with some of the cosmogonic ideas of Ambartsumyan [5, 7] which are, in fact, logical generalizations of a large volume of observational material.

Not all aspects of the modified theory of gravitation have been adequately developed and—this is particularly important—the theory has not as yet reached a stage where it could be compared with astronomical facts. It is therefore essential to derive new effects which are not predicted by the Newton-Einstein theory of gravitation. We must emphasize that the theory is purely speculative and we shall be able to discuss its validity only after the predicted effects have been compared with observational data.

In our previous paper [8, 10] we investigated the Newtonian variant of the generalized theory of gravitation. The field was characterized by two scalar functions, namely, the potential φ and the gravitational scalar k (i.e., the gravitational "constant" which in the generalized theory is allowed to vary). We calculated the parameters of static spherical configurations consisting of an incompressible fluid, a degenerate neutron gas, and a degenerate electron-nucleon gas. It was found that, with a reasonable choice of the boundary conditions, the gravitation scalar is always zero at the center of the configurations, and then monotonically increases with distance from the center, tending to the Newtonian value at infinity. Next, the curve representing the mass M as a function of the central density ρ_0 consists of two branches, i.e., the function $M(\rho_0)$ is two-valued. The lower branch, called the normal branch, is not very different from the corresponding $M(\rho_0)$ curve obtained from the usual theory of gravitation. The other, i.e., the anomalous branch, represents the baryon and electron configurations with masses from very low values up to masses of the order of galactic mass and more. The radii of these bodies are less than their gravitational radii* and, therefore, these results must be verified on the basis of the relativity theory even from the qualitative point of view.

This is the first of a series of papers on the generalized Einstein theory of gravitation. Following Dirac and Jordan, we shall assume that the gravitational "constant" is, in fact, a function of space-time coordinates. It is clear that any modification of the existing theory of gravitation must be consistent with facts known from celestial mechanics. The correspondence principle will be discussed in detail in our next paper.

2. In the generalized theory of gravitation the field is characterized by 11 independent functions, namely, the components of the metric tensor g_{ik} and the scalar $\varkappa = 8\pi k/c^2$. The theory is based on Jordan's variational principle

$$S = \frac{c}{2}\int \varkappa^{\eta}\left(R + \frac{2\varkappa}{c^2}\Lambda + \zeta g^{ik}\frac{\varkappa_i \varkappa_k}{\varkappa^2}\right)\sqrt{-g}\, d\Omega, \qquad (1)$$

where R is the scalar curvature of space, Λ is the Lagrange density of matter, and $\varkappa_i = \partial\varkappa/\partial x^i$, η and ζ are dimensionless constants which are the parameters of the new theory. The above expression for the action S has a number of well-known standard properties, but it also satisfies the following requirements: when $\varkappa =$ const it becomes identical with the corresponding expression in the Einstein gravitation theory and, moreover, it does not contain any new dimensional constants. The square of the four-dimensional interval will be written in the form

$$ds^2 = -g_{ik}\, dx^i dx^k. \qquad (2)$$

The first variation of the action is given by

$$\delta S = \int \varkappa^{\eta}\left\{\left[R_{ik} - \frac{1}{2}Rg_{ik} - \frac{\varkappa}{c^2}T_{ik} + \right.\right.$$
$$+ (\zeta - \eta(\eta - 1))\frac{\varkappa_i \varkappa_k}{\varkappa^2} - \eta\frac{\varkappa_{i;k}}{\varkappa} + \ldots$$

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*We shall preserve this term for the quantity $r_g = 2M$ although even in the relativistic variant of the generalized theory there is no Schwarzschild type singularity.

$$+\eta g_{ik}\frac{\varkappa^l_{;l}}{\varkappa}-\frac{\zeta-2\eta(\eta-1)}{2}g_{ik}\frac{\varkappa_l\varkappa^l}{\varkappa^2}\Big]\delta g^{ik}+$$

$$+\frac{\eta}{\varkappa}\Big[R+\frac{2(\eta+1)}{\eta c^2}\varkappa\Lambda-$$

$$-\frac{\zeta(\eta-2)}{\eta}\frac{\varkappa_l\varkappa^l}{\varkappa^2}-\frac{2\zeta}{\eta}\frac{\varkappa^l_{;l}}{\varkappa^2}\Big]\delta\varkappa\Big\}\sqrt{-g}\,d\Omega=0, \quad (3)$$

where the semicolon represents covariant differentiation, $\varkappa^l = g^{lk}\varkappa_k$, and the energy-momentum tensor is related to the Lagrange density Λ by the formula

$$T_{ik}=-\frac{2}{\sqrt{-g}}\times$$

$$\times\left[\frac{\partial(\sqrt{-g}\Lambda)}{\partial g^{ik}}-\frac{\partial}{\partial x^l}\frac{\partial(\sqrt{-g}\Lambda)}{\partial g^{ik}_{;l}}\right].$$

For a continuous medium, and in the absence of dissipative processes,

$$T_{ik}=(P+\rho)u_i u_k+Pg_{ik}, \quad (4)$$

where P is the pressure, ρ is the energy density, and u_i is the four-dimensional velocity.

From Eq. (2) we obtain the following set of equations:

$$[\zeta-\eta(\eta-1)]\frac{\varkappa_i\varkappa_k}{\varkappa^2}-\eta\frac{\varkappa_{i;k}}{\varkappa}+$$

$$+g_{ik}\left[\eta\frac{\varkappa^l_{;l}}{\varkappa}-\frac{\zeta-2\eta(\eta-1)}{2}\frac{\varkappa_l\varkappa^l}{\varkappa^2}\right]+$$

$$+R_{ik}-\frac{1}{2}Rg_{ik}=\frac{\varkappa}{c^2}T_{ik}, \quad (5)$$

$$R+\frac{2(\eta+1)}{\eta c^2}\varkappa\Lambda-$$

$$-\frac{\zeta(\eta-2)}{\eta}\frac{\varkappa_l\varkappa^l}{\varkappa^2}-\frac{2\zeta}{\eta}\frac{\varkappa^l_{;l}}{\varkappa}=0. \quad (6)$$

The above equations can readily be reduced to a more convenient form. To do this, let us multiply the first of them by g^{ik} and sum over repeated indices. Using the second equation, we can then show that

$$\frac{\varkappa^l_{;l}}{\varkappa}+(\eta-1)\frac{\varkappa_l\varkappa^l}{\varkappa^2}=$$

$$=\frac{\eta\varkappa}{c^2(3\eta^2-2\zeta)}\left(T-2\frac{\eta+1}{\eta}\Lambda\right). \quad (7)$$

Let us now multiply Eq. (6) by $g_{ik}/2$, add it to Eq. (5), and then subtract Eq. (7). The result is

$$R_{ik}-\eta\frac{\varkappa_{i;k}}{\varkappa}+[\zeta-\eta(\eta-1)]\frac{\varkappa_i\varkappa_k}{\varkappa^2}=$$

$$=\frac{\varkappa}{c^2}\left(T_{ik}+\frac{\zeta-\eta^2}{3\eta^2-2\zeta}Tg_{ik}\right)-\frac{\eta(\eta+1)\varkappa\Lambda g_{ik}}{c^2(3\eta^2-2\zeta)}. \quad (8)$$

To transform to the Jordan notation [1] we must introduce the following replacements in the above formulas:

$$g_{ik}\to g_{ik},\quad \varkappa_k\to\varkappa_k,$$

$$\Lambda\to-\Lambda,\quad R\to R,\quad T\to -T.$$

We note that in the variational principle given by Eq. (1) we can replace R by

$$G=g^{ik}\left[(\Gamma^l_{ik}\Gamma^m_{lm}-\Gamma^m_{il}\Gamma^l_{km})+\frac{\eta}{\varkappa}(\Gamma^l_{ik}\varkappa_l-\Gamma^m_{im}\varkappa_k)\right], \quad (9)$$

which does not contain the second derivatives of g_{ik}. Although G is not a scalar, the quantity $\int\varkappa^\eta G\sqrt{-g}\,d\Omega$ is a scalar and is equal to $\int\varkappa^\eta R\sqrt{-g}\,d\Omega$.

3. The equations of motion can be obtained from the following variational principle:

$$\delta S_m=\delta\int\varkappa^{\eta+1}\Lambda\sqrt{-g}\,d\Omega=0. \quad (10)$$

Here, we must vary the particle trajectory

$$x'^k=x^k+\xi^k,$$

where ξ^k are infinitely small quantities. For δg^{ik} and $\delta\varkappa$ we then obtain

$$\delta g^{ik}=\xi^{i;k}+\xi^{k;i},\qquad \delta\varkappa=-\varkappa_k\xi^k. \quad (11)$$

Using Eqs. (10) and (11), we obtain

$$\delta S_m=\int[-(\eta+1)\varkappa^\eta\sqrt{-g}\,\Lambda\varkappa_l\xi^l+$$

$$+\varkappa^{\eta+1}\delta(\sqrt{-g}\,\Lambda)]\,d\Omega=$$

$$=\int[(\varkappa^{\eta+1}T^k_l)_{;k}-(\eta+1)\varkappa^\eta\Lambda\varkappa_l]\,\xi^l\sqrt{-g}\,d\Omega=0.$$

Since the variations in the coordinates ξ^l are arbitrary, it follows that

$$(\varkappa^{\eta+1}T^k_i)_{;k}=(\eta+1)\varkappa^\eta\Lambda\varkappa_i. \quad (12)$$

This is, in fact, the required equation of hydrodynamics.

The relation given by Eq. (12) can also be obtained directly from the field equations. To show this, let us first raise the index k in Eq. (5), multiply by $\varkappa^\eta$, and evaluate the covariant derivative with respect to k. Next, let us multiply Eq. (6) by $0.5\,\eta\varkappa^{\eta-1}\varkappa_k\delta^k_i$, and add the resulting equation to the previous one, thus obtaining Eq. (12).

In the presence of the electromagnetic field we must augment the integrand in Eq. (1) by the following two terms:

$$\frac{2\varkappa}{c^3}A_k j^k-\frac{\varkappa}{8\pi c^2}F_{ik}F^{ik},$$

where j^k is the current density, A_k is the four-dimensional potential, and $F_{ik} = A_{k;i} - A_{i;k}$. To derive the equations for the electromagnetic field we must fix all quantities characterizing the gravitational field and the particle trajectory, and subject the potentials A_i to virtual variations. The final result is the second pair of Maxwell's equations:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^k}\left(\sqrt{-g}\,\varkappa^{\eta+1}F^{ik}\right)=\frac{4\pi}{c}\varkappa^{\eta+1}j^i .$$

Hence we have the continuity equation

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\,\varkappa^{\eta+1}j^i\right)=0.$$

Thus, the new theory of gravitation contains two dimensionless parameters, namely, η and ζ. By considering the three effects in the general theory of relativity, and taking into account observational results, it is possible to find one condition which relates them. It is therefore necessary to introduce some further considerations in order to determine the numerical value of one of them.

From Eq. (12) and from the continuity equation given above, it is clear that in the new theory the laws of conservation of energy and of electric charge are not satisfied for $\eta \neq -1$ in the form in which they are known at present. If, however, we substitute $\eta = -1$ we obtain the usual conservation laws

$$T^k_{i;k}=0,\qquad \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\,j^i\right)=0.$$

Henceforth we shall take $\eta = -1$. Equations (7) and (8) will then assume the following form:

$$\varkappa^l_{;l}-2\frac{\varkappa_l\varkappa^l}{\varkappa^2}=\frac{\varkappa T}{c^2(2\zeta-3)}, \qquad (14)$$

$$R^k_i+\frac{\varkappa^k_{;i}}{\varkappa}+(\zeta-2)\frac{\varkappa_i\varkappa^k}{\varkappa}=\frac{\varkappa}{c^2}\left(T^k_i+\frac{\zeta-1}{3-2\zeta}T\delta^k_i\right). \qquad (15)$$

Moreover,

$$R=\frac{2\zeta}{c^2(3-2\zeta)}\varkappa T-\zeta\frac{\varkappa_l\varkappa^l}{\varkappa^2}, \qquad (16)$$

which is obtained from Eqs. (14) and (15).

4. Consider the static gravitational field with spherical symmetry. The square of the four-dimensional interval will be written in the form

$$ds^2=c^2e^\nu dt^2-e^\lambda dr^2-r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right), \qquad (17)$$

where ν, λ are unknown functions of R. From Eq. (4) we have

$$T^1_1=T^2_2=T^3_3=P,\qquad T^0_0=-\rho. \qquad (18)$$

For nonzero components of the tensor R^k_i we have

$$R^0_0=e^{-\lambda}\left(\frac{1}{4}\nu'\lambda'-\frac{\nu'}{r}-\frac{1}{2}\nu''-\frac{1}{4}\nu'^2\right),$$

$$R^1_1=e^{-\lambda}\left(\frac{1}{4}\nu'\lambda'+\frac{\lambda'}{r}-\frac{1}{2}\nu''-\frac{1}{4}\nu'^2\right),$$

$$R^2_2=R^3_3=e^{-\lambda}\left(\frac{1}{2r}\lambda'-\frac{\nu'}{2r}-\frac{1}{r^2}\right)+\frac{1}{r^2}.$$

Using these expressions and Eq. (18), we find from Eqs. (14) and (15) that

$$\frac{\varkappa''}{\varkappa}+\frac{\varkappa'(\nu'-\lambda')}{2\varkappa}+\frac{2\varkappa'}{r\varkappa}-\frac{2\varkappa'^2}{\varkappa^2}=\frac{\varkappa(\rho-3P)}{c^2(3-2\zeta)}e^\lambda,$$

$$\frac{\nu'\lambda'}{4}-\frac{\nu'}{r}-\frac{\nu''}{2}-\frac{\nu'^2}{4}+\frac{\varkappa'\nu'}{2\varkappa}=$$

$$=\frac{(\zeta-2)\rho+3(\zeta-1)P}{c^2(3-2\zeta)}\varkappa e^\lambda,$$

$$\frac{\nu'\lambda'}{2r}+\frac{\lambda'}{r}-\frac{\nu''}{2}-\frac{\nu'^2}{4}+\frac{\varkappa''}{\varkappa}-\frac{\varkappa'\lambda'}{2\varkappa}+$$

$$+(\zeta-2)\frac{\varkappa'^2}{\varkappa^2}=\frac{\zeta P+(1-\zeta)\rho}{c^2(3-2\zeta)}\varkappa e^\lambda,$$

$$\frac{\lambda'-\nu'}{2r}+\frac{1}{r^2}(e^\lambda-1)+$$

$$+\frac{\varkappa'}{r\varkappa}=\frac{\zeta P+(1-\zeta)\rho}{c^2(3-2\zeta)}\varkappa e^\lambda. \qquad (19)$$

Next, it follows from Eq. (13) that

$$P'+\frac{P+\rho}{2}\nu'=0. \qquad (20)$$

We thus have five equations for the four unknown functions $\varkappa$, ν, λ and P. One of the equations is a consequence of the remaining three and can be rejected. Relatively simple equations can be obtained if the second and third equations in Eq. (12) are replaced by their difference, and instead of the first equation we take the combination I − II + III + IV, leaving the last equation unaltered. This yields the following set of four equations:

$$\frac{\varkappa''}{\varkappa}+\frac{\zeta-4}{2}\frac{\varkappa'^2}{\varkappa^2}+\frac{2\varkappa-r\varkappa'}{2r\varkappa}\lambda'+$$

$$+\frac{2\varkappa'}{r\varkappa}+\frac{1}{r^2}(e^\lambda-1)=\frac{1}{c^2}\varkappa\rho e^\lambda,$$

$$\frac{\lambda'}{2r}-\frac{\nu'}{2r}+\frac{\varkappa'}{r\varkappa}+\frac{1}{r^2}(e^\lambda-1)=$$

$$=\frac{\zeta P+(1-\zeta)\rho}{c^2(3-2\zeta)}\varkappa e^\lambda,$$

$$\frac{2\varkappa-r\varkappa'}{2r\varkappa}\nu'+\zeta\frac{\varkappa'^2}{2\varkappa^2}-\frac{2\varkappa'}{r\varkappa}-\frac{1}{r^2}(e^\lambda-1)=\frac{1}{c^2}\varkappa Pe^\lambda,$$

$$P'+\frac{P+\rho}{2}\nu'=0. \qquad (21)$$

It will be convenient to use a set of units in which $c = k_0 = 1$, $m_n^4c^5/(32\pi^2\hbar^3) = 1/4\pi$ where k_0 is the "usual gravitational constant, m_n is the neutron mass, and $\hbar$ is Planck's constant divided by 2π. Eliminating ν' from Eq. (21), and substituting $k = \varkappa/8\pi$, we have after some simple transformations

$$\frac{P'}{P+\rho}=-\frac{r}{2-r\dfrac{k'}{k}}\times$$

$$\times\left[8\pi kPe^\lambda+\frac{2}{r}\frac{k'}{k}-\frac{\zeta}{2}\left(\frac{k'}{k}\right)^2+\frac{e^\lambda-1}{r^2}\right],$$

$$\lambda' = 16\pi kre^{\lambda}\left[\frac{\zeta P + (1-\zeta)\rho}{3-2\zeta} + \frac{P}{2 - r\frac{k'}{k}}\right] -$$

$$-2\frac{e^{\lambda}-1}{r}\cdot\frac{1 - r\frac{k'}{k}}{2 - r\frac{k'}{k}} + (2-\zeta)\frac{r\left(\frac{k'}{k}\right)^2}{2 - r\frac{k'}{k}},$$

$$\left(\frac{k'}{k}\right)' = 8\pi ke^{\lambda}\left\{(\rho - P) + \right.$$

$$\left.+\frac{r\frac{k'}{k} - 2}{3-2\zeta}[\zeta P + (1-\zeta)\rho]\right\} - \frac{k'}{k}\frac{e^{\lambda}+1}{r}. \quad (22)$$

To obtain a correct solution of the above internal problem, we must augment these equations by the condition of thermal equilibrium, and specify the source distribution function and the energy transfer coefficients (dissipation processes must then also be taken into account in Eq. (4) for the energy-momentum tensor). However, our intention is to consider the more modest problem, namely the development of models for stars consisting of degenerate gaseous masses. All that then remains is to augment Eq. (22) only by the equation of state for cold matter $P = P(\rho)$.

As the boundary conditions we can obviously take the requirement that P vanish and the functions λ, k, and k' be continuous on the surface of the configuration.

5. The external solution of the field equation is given in [1]. It is known as the Heckmann solution and is of the form

$$r = \frac{r_0}{\sqrt{\tau}\,(\tau^{-h} - \tau^{h})},$$

$$e^{-\lambda/2} = \frac{1}{2h}\left|\left(\frac{1}{2}+h\right)\tau^{h} - \left(\frac{1}{2}-h\right)\tau^{-h}\right|,$$

$$e^{\nu} = \tau^{1/(1+2\beta_0)},$$

$$k = k_0\,\tau^{-3\beta_0/(1+2\beta_0)}, \quad (23)$$

where τ is a variable parameter, k_0, r_0, and β_0 are integration constants, and

$$h^2 = \frac{1}{4} - \frac{\beta_0}{2}\frac{1+\beta_0\zeta}{(1+2\beta_0)^2}. \quad (24)$$

When $\beta_0 = 0$, $h^2 = 1/4$, Eq. (23) becomes identical with the Schwarzschild equation. For negative β_0 we have $h^2 > 1/4$ while for positive β_0 we have $h^2 < 1/4$. Here we shall investigate the case $\beta_0 < 0$, $h > 1/2$ (Heckmann's solution invariant under the replacement of h by $-h$). When $r_0 > 0$ we have $0 < \tau < 1$, and the function $r(\tau)$ increases monotonically: $r(0) = 0$ and $r(1) = \infty$. When $\tau \to 1$ the metric becomes Euclidean asymptotically, i.e., $r \to \infty$, $e^{\lambda} \to e^{\nu} \to 1$, $k(r) \to k_0$. The requirement that the generalized theory of gravitation become identical with the ordinary theory for sufficiently large distances yields $k_0 = 1$.

Comparison of the Heckmann and Schwarzschild solutions at large distances gives

$$r_0 = 4hM(1+2\beta_0). \quad (25)$$

By considering the limiting case it is also possible to determine the constant β_0. This can be done by comparing the solution of Eq. (14) for $\rho \to 0$ with the Heckmann solution at large distances. In this way, Jordan has found that

$$\beta_0 = \frac{1}{2\zeta - 3}. \quad (26)$$

We are thus left with only one undetermined parameter, namely ζ, whose value can be found only by comparing the new theory with experiment. At present there are three known general relativity effects, namely, the deflection of light, the red shift, and the precession of the perihelion of planetary orbits. These are well confirmed by observations. These effects have also been investigated in the generalized theory of gravitation [1]. It turns out that the red shift introduces nothing new, but the other two effects impose a definite limitation on the absolute value of ζ. Thus, if it is required that the angular displacement of the perihelion of the orbit of a test body rotating about some other massive celestial body must agree to within 2% with the general relativity results for a constant $\varkappa$ (this is the accuracy with which the precession of the perihelion of Mercury has been measured), we obtain the following condition: $|4\beta_0/3| \leq 0.02$. In the ensuing calculation we shall assume that

$$\zeta = -30,$$

which corresponds to $4\beta_0/3 \approx -0.02$.

The external solution includes the mass M of the celestial body. This is the so-called active mass, which is measured by the observer in terms of its gravitational effect at sufficiently large distances, where Newton's law is valid. To determine this mass we shall start with the Gauss theorem which is valid for the static field:

$$\int R_0^0\sqrt{-g}\,dV = -4\pi M.$$

From Eq. (15) we then have

$$R_0^0 = 8\pi k\left(T_0^0 + \frac{\zeta - 1}{3-2\zeta}T\right) - \frac{k'\nu'}{2k}.$$

Substituting this expression into the Gauss formula, we have

$$M = -2\int k\left(T_0^0 + \frac{\zeta-1}{3-2\zeta}T\right)\sqrt{-g}\,dV +$$

$$+\frac{1}{8\pi}\int\frac{k'\nu'}{k}\sqrt{-g}\,dV. \quad (27)$$

For a spherically symmetric distribution of matter we have

$$M = \frac{2}{3-2\zeta}\int k[(3-\zeta)\rho +$$

$$+3(1-\zeta)P]\sqrt{-g}\,dV + \frac{1}{8\pi}\int\frac{k'\nu'}{k}\sqrt{-g}\,dV. \quad (28)$$

We can now use the external solution and the above definition of the concept of mass M to formulate the boundary conditions on the surface of the configuration:

$$P(R) = 0,$$

$$R = \frac{4hBM}{\sqrt{\tau_0(\tau_0^{-h} - \tau_0^{h})}},$$

$$\lambda(R) = 2\ln 2h - 2\ln\left[(h + 0.5)\tau_0^{h} + (h - 0.5)\tau_0^{-h}\right],$$

$$k(R) = \tau_0^{1/(1-2\zeta)},$$

$$k'(R) = \frac{2M}{3 - 2\zeta}\,\frac{1}{R^2}\,e^{\lambda(R)/2}\,\tau_0^{-(2\zeta+1)/2(2\zeta-1)}, \tag{29}$$

where R is the radius of the configuration, $\tau_0 \equiv \tau(R)$, and

$$h = (4\zeta^2 - 10\zeta + 7)^{1/2}/2(1 - 2\zeta),$$

$$B = (2\zeta - 1)/(2\zeta - 3). \tag{30}$$

For the numerical values of these quantities we have

$$\zeta = -30, \qquad h = 0.51234, \qquad B = 0.96825.$$

It is important to note the following exceedingly important point: in the Einstein variant of the general theory of gravitation the metric does not have the singularities which are present in the Schwarzschild solution (this is clear from Eq. (23)). This striking result enables us to develop models of static superdense configurations with exceedingly high masses.

Let us now investigate the limiting transition to the Einstein theory of gravitation with constant k. We note from the variational principle given by Eq. (1) that this transition can be carried out by substituting $|1/\zeta| \to 0$. The external solution given by Eq. (23) for a point mass must become identical with the Schwarzschild solution. Cases of positive and negative values of ζ must be discussed separately. We recall that we are confining our attention to $\zeta = -30$. Let us now consider Eq. (23) and take into account the numerical constants given by Eq. (30). It is readily noted that the $e^{\lambda(r)}$ is zero for r = 0, but increases monotonically with increasing r, reaching a maximum in the neighborhood of r_0 after which it falls asymptotically to $e^{\lambda(\infty)} = 1$. As $|\zeta|$ increases the maximum of e^{λ} increases and becomes infinite for $\zeta \to -\infty$. The quantity r_0, on the other hand, approaches the gravitational radius $r_g = 2M$ and becomes equal to it in the limit. When $r \geq r_0$ the asymptotic behavior of $e^{\lambda(r)}$ is given by the Schwarzschild solution $e^{\lambda} = 1/(1 - r_g/r)$. Inside the gravitational radius, however, we have $e^{\lambda(r)} \to 0$ at each point. Finally, the function $e^{\lambda(r)}$ increases monotonically for finite values of ζ; it becomes equal to $1 - r_g/r$ outside the gravitational radius and to zero inside this radius when $\zeta \to -\infty$.

Let us now express the gravitational scalar k in terms of r by eliminating the parameter τ. Assuming that $|\zeta| \gg 1$, we find that

$$r_0/r \approx k^{-3/4} - k^{-2\zeta}.$$

If we now allow $-\zeta$ to tend to infinity, we obtain in the limit $k(r) = (r/r_0)^{4/3}$ for $r \leq r_g$ and $k(r) = 1$ for $r \geq r_g$. For each finite ζ, the function k(r) takes the form of a monotonically increasing smooth function.

We recall that in the Newtonian variant of the generalized theory of gravitation the range of distances where k(r) is substantially different from a constant contracts to a point as $\zeta \to -\infty$.

At the same time, if the solution given by Eq. (23) outside the gravitational radius goes over into the Schwarzschild solution, then inside the gravitational sphere the character of this solution does not correspond to the predictions of the general theory of relativity.

6. Preliminary analysis of the problem shows that in the relativistic variant of the generalized theory of gravitation there are both configurations with masses of the order of the solar mass and static superdense formations whose masses exceed the solar mass by many orders of magnitude. Models of such hypothetical bodies will be called gravitars, emphasizing by this that their properties are due to the variability of the gravitational scalar k(r). The radii of the gravitars are of the order of, or less than, their gravitational radius $R_g = 2M$.

Gravitars have an exceedingly high mass defect: $\Delta M = M_0 - M$, where M is the active gravitational mass defined by Eq. (28), and M_0 is the proper mass, i.e., the mass of the material without taking the energy of gravitational interaction into account:

$$M_0 = 4\pi\int \rho e^{\lambda/2} r^2 dr. \tag{31}$$

This is explained by the fact that inside and on the surface of gravitars the functions k(r), $e^{\lambda(r)}$ and $e^{\nu(r)}$ are small in comparison with unity, and $M/M_0 \approx ke^{\nu/2} \ll 1$.

The gravitational red shift for gravitars may turn out to be of the order of the proper frequency of the emitted light. It is given by the equation

$$\frac{\Delta\omega}{\omega} = \sqrt{g_{00}(R)} - 1.$$

From Eq. (29) we find that

$$\sqrt{g_{00}(R)} = e^{\lambda(R)/2} \approx \tau_0^{1/2B}.$$

Gravitars are characterized by small values of the parameter $\tau_0 \ll 1$, and therefore

$$\frac{\Delta\omega}{\omega} \approx -1.$$

Since they are very massive, gravitars are relatively compact in the sense that for them $w = M/R \gg 1/2$. The strong gravitational field on the surface of gravitars gives rise to large deflections of the light emitted by them. Therefore, a distant observer will see them as much larger objects than they really are.

In future papers we shall integrate Eq. (22) subject to the conditions given by Eq. (29) for different equations of state of the material.

We are indebted to Academician V. A. Ambartsumyan and our colleagues V. A. Vardanyan, Yu. L. Vartanyan, N. B. Engibaryan, R. S. Oganesyan, D. M. Sedrakyan, and E. V. Chubaryan for their constant interest in the work and many discussions. Preliminary calculations were performed on the Nairi computers at the Computational Center of the Academy of Sciences of the Armenian SSR, at the Byurakan Observatory, and at its branch. We are grateful to the directors of the these laboratories for enabling us to perform this work.

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17 April 1968

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