On the reduction of the radiative transfer problem in a finite layer to the problem for a half-space

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1. Introduction. The contemporary development of radiative transfer theory owes much to the method, introduced by Ambartsumyan, of the addition of layers.[1-3] This method was effectively applied first to the solution of linear problems in homogeneous plane-parallel layers, and then attempts were made to extend it to more complex problems – to an inhomogeneous medium, spherical shapes, and even nonlinear processes.

The idea of the method of addition of layers is as follows. To a layer of thickness τ_0, the properties of which are of interest to us, we add another layer of thickness τ and establish functional relationships between the optical properties of these two layers and the total layer.

Usually the added layer τ is chosen infinitesimally thin. In the first place, the properties of an infinitesimally thin layer are well known. They are given by the local properties of the specific medium. In addition, its contribution to the transfer process is fairly simple to take into account, because in it the scattering of higher than first order can be ignored. In the second place, the functional equations themselves in this case are simplified, being transformed into differential equations with respect to τ_0.

Thus, in practice the method of addition of layers implies adding to the layer in question an infinitesimally thin layer and is called the Ambartsumyan invariance principle.

The problem for a semiinfinite medium is a particular case of the problem of a finite layer. In this case the equations are greatly simplified, because the thickness of the layer τ_0 drops out of consideration. As a result of this the solution to many problems for a half-space can be obtained in explicit form.

In connection with the fact that the problem for a half-space is much simpler to solve, it is natural to try to reduce the solution of the problem of a finite layer to the solution of the corresponding problem in a semi-infinite medium.

It is shown below that in principle it is completely possible to connect the solutions of the problems in finite and semiinfinite layers. For this we do not add to the finite layer in question an infinitesimally thin one, but rather an infinitely thick one (!). In so doing, we make use of the fact that both the added layer and the total layer are semiinfinite.

The results obtained below in Secs. 2 and 3 are applicable in the most general case, i.e., for any scattering indicatrix and for every given linear law of redistribution over frequencies.

2. The relaxation between solutions in finite and semiinfinite layers. Let a certain distribution f_0 (over frequencies, angles, etc.) emerge from a semiinfinite medium. We put a layer of thickness τ in front of the medium. Then, the primary radiation undergoes additional acts of absorption and scattering in the total semiinfinite layer, and as a result another distribution f_τ emerges from it. We define the transformation of the primary radiation by the added layer τ by the action of some linear operator $Y(\tau)$: $f_\tau = Y(\tau)f_0$. We emphasize that $Y(\tau)$ gives the transformation of that radiation which at the depth τ is directed so as to emerge from the semiinfinite medium.

We also introduce the operator $Z(\tau) = Y(\tau)R_\infty$, where R_∞ is the reflection operator from the semiinfinite layer. If there exists primary radiation in the semiinfinite medium at the depth τ which is directed toward the interior of the medium, then the emerging radiation is determined by the action of $Z(\tau)$.

In a layer of finite thickness τ_0 let a source density $g(\tau)$ be given. We are interested in the radiation of this layer to the right i^- and to the left i^+. It is understood that i^- exchanges meaning with i^+ when $g(\tau_0 - \tau)$ is substituted for $g(\tau)$.

We add on the right of the layer in question with the sources a semiinfinite layer (without sources). Let J^+ be the radiation emerging from the total semiinfinite layer; we consider this to be known. It is composed of radiation to the left by the finite layer directly and the primary radiation of the finite layer to the right with the eventual output $Z(\tau_0)$:

$$J^+ = i^+ + Z(\tau_0)i^-, \quad J^- = i^- + Z(\tau_0)i^+. \tag{1}$$

The second equation is obtained by adding a semiinfinite layer on the left or by replacing τ with $\tau_0 - \tau$. This is just the connection sought.

We introduce the quantities $Y(\tau, \tau_0)$ and $Z(\tau, \tau_0)$ analogous to $Y(\tau)$ and $Z(\tau)$, for a layer of finite thickness τ_0. Adding to this layer a semiinfinite layer on the right and on the left, we obtain

$$\begin{aligned} Y(\tau) &= Y(\tau, \tau_0) + Z(\tau_0)Z(\tau, \tau_0), \\ Z(\tau_0 - \tau) &= Z(\tau, \tau_0) + Z(\tau_0)Y(\tau, \tau_0). \end{aligned} \tag{2}$$

Recall that Y and Z do not depend on the internal sources and, according to the irreversibility principle for optical phenomena, describe the internal conditions when the layer is illuminated by a parallel beam.

In the particular case when a photon is absorbed in layer of thickness τ_0 at a depth τ, relations (1) determine the probability $p(\tau, \tau_0)$ of its emerging in terms of the probability $P(\tau)$ of emerging from a depth τ of a semi-infinite medium:

$$P(\tau)=p(\tau, \tau_0)+Z(\tau_0)p(\tau_0-\tau, \tau_0), \quad P(\tau_0-\tau)=p(\tau_0-\tau, \tau_0)+Z(\tau_0)p(\tau, \tau_0). \tag{3}$$

In the very particular case of $\tau = 0$ there follows from (3) linear equations for the Ambartsumyan functions $\varphi(\tau_0)$ and $\psi(\tau_0)$.

Relations (2) for $\tau_0 = \tau$, on the other hand, establish a relation[5] between the reflection and transmission operators $R(\tau)$ and $Q(\tau)$ of a finite layer and the operators R_∞ and $Y(\tau)$:

$$Y=Q+ZR, \quad R_\infty=R+ZQ. \tag{4}$$

We see that the linear systems of equations for every problem for a finite layer are essentially identical. Then, by addition and subtraction we arrive at separate equations for the sum and difference of the unknown quantities. In the general case these are linear integral equations with a continuous operator kernel Z. They are especially convenient for numerical solution by discretization. Further investigation of the equations is based on the properties of the operator Z.

3. Properties of the operators Y and Z. By putting successive layers with thicknesses τ_1 and τ_2 in front of the semiinfinite medium, we find[5]

$$Y(\tau_1+\tau_2)=Y(\tau_1)Y(\tau_2). \tag{5}$$

This very important semigroup property of Y is equivalent to the equations

$$-dY(\tau)/d\tau=GY=YG, \quad Y(0)=I, \tag{6}$$

where I is the unit operator, and $G = -Y'(0)$.

The operator G describes the change in the distribution emerging from the semiinfinite medium when an infinitesimally thin layer is placed in front of it. It can be expressed simply in terms of the Ambartsumyan operator φ and is easily constructed from physical considerations according to the definition $Y(d\tau) = I - Gd\tau$. According to (6) Y is, in general, a superposition of solutions $\tilde{Y}_k$ of the problem $G\tilde{Y}_k = k\tilde{Y}_k$.

We see that the problem for a half-space is in essence the problem of the eigenfunctions and eigenvalues of the operator G. Note that $\tilde{Y}_k$ is the solution to the Milne problem with a discrete spectrum $k < 1$ for the operator G (the asymptotic behavior in the deep layers of the half-space).

It is clear that $Z(\tau_1 + \tau_2) = Y(\tau_1)Z(\tau_2)$ and $P(\tau_1 + \tau_2) = Y(\tau_1)P(\tau_2)$, and that we have the property, useful for investigating asymptotic behavior, $P(\tau) = (1/2)\lambda[Y(\tau) + Z(\tau)]p$, where p is the indicatrix.

The obvious property $(1/2)\lambda Y(\tau)\varphi = P(\tau)$ enables us to get from (6) an integral representation for $Y(\tau)$ in terms of $P(\tau)$ and analogously for $Z = YR_\infty$. This means that the kernel of the integral equations (1)-(4) can be expressed in terms of the solution of $P(\tau)$ for a half-space.

In regard to the possibility of explicitly reducing the solution of the problem in a finite layer to the corresponding solution for a semiinfinite medium, we shall treat this in specific examples.

4. Particular problems. In another paper[6] we have illustrated the proposed approach in a one-dimensional medium with monochromatic scattering. In this case $G \equiv k$ and R_∞ are constants, and from (6) it follows that $Y(\tau) = e^{-k\tau}$ and $Z(\tau) = R_\infty e^{-k\tau}$. Equations (1)-(4) are transformed into linear algebraic equations. For example, (4) leads immediately to the solution (cf. refs. 2 and 3)

$$R(\tau)=R_\infty\frac{1-Y^2(\tau)}{1-Z^2(\tau)}, \quad Q(\tau)=\frac{1-R_\infty^2}{1-Z^2(\tau)}Y(\tau) \tag{7}$$

for the reflection and transmission coefficients for a layer of thickness τ. This is one example of the explicit reduction of the solution for a finite problem to the solution for a semiinfinite problem.

The monochromatic scattering problem in a three-dimensional medium with a spherical indicatrix is treated by Danielyan and Mnatsakanyan.[7] In this case it is easy to see that

$$G(\eta,\zeta)=\frac{\delta(\eta-\zeta)}{\zeta}-\frac{\lambda}{2}\frac{\varphi(\eta)}{\zeta}, \tag{8}$$

and the commutativity condition (6) $GY = YG$ leads to an expression for $Y(\tau, \eta, \xi)$, and hence also for $Z(\tau, \eta, \xi)$ in terms of the probability of emergence $P(\tau, \eta)$:

$$Z(\tau,\eta,\zeta)=\frac{\lambda}{2}\eta\varphi(\eta)\frac{F(\tau,\eta)+\bar{F}(\tau,\zeta)}{\eta+\zeta}; \tag{9}$$

here

$$F(\tau,\eta)=\frac{P(\tau,\eta)}{P(0,\eta)}, \quad \bar{F}(\tau,\zeta)=\zeta\varphi(\zeta)\int_0^1\frac{P(\tau,\mu)}{\mu+\zeta}d\mu.$$

We see that $Z(\tau, \eta, \xi)$ is expressed in terms of functions of two arguments. Mathematically this result was derived earlier (in solving a certain class of integral equations) and is published in a paper of Engibaryan and the present author.

Equation (3) can be transformed to the form

$$a(\eta)p(\tau,\tau_0,\eta)-\frac{\lambda}{2}\eta\int_0^1\frac{A(\tau_0,\mu,\eta)}{\mu-\eta}$$

$$\times p(\tau,\tau_0,\mu)d\mu=\frac{\lambda}{2}e^{-\tau/\eta}A(\tau_0-\tau,0,\eta),$$

$$a(\eta)=1-\frac{\lambda}{2}\eta\ln\frac{1+\eta}{1-\eta}, \quad A(\tau,\mu,\eta)=1+e^{-\tau/\eta}\frac{\bar{F}(\tau,\mu)-\bar{F}(\tau,\eta)}{\varphi(\eta)}, \tag{10}$$

which is obtained by successive application to (3) and (9) of the operators $\int_0^1\ldots d\eta/(\eta\pm\zeta)$ after a number of algebraic manipulations using the singular equation for $P(\tau, \eta)$ of the form (10) with $\tau_0 = \infty$ (Danielyan). Some additional considerations of a physical nature give us reason to hope to find the solution for $p(\tau, \tau_0, \eta)$ in terms of the solution of the corresponding semiinfinite problem.

Another example of explicit but approximate reduction of the solution to the problem in a finite layer to the problem for a half-space is for all practical purposes provided by the asymptotic solutions of Sobolev for a layer of finite but large optical thickness $\tau_0 \gg 1$ (refs. 3, 4, 8). In practice they are applicable starting from $\tau_0 \approx 2$. It would be desirable to have solutions suitable also for small τ_0.

It turns out that in the general case, beginning from Eqs. (1)-(4), we can get stronger asymptotic solutions practically applicable for a layer of any thickness $\tau_0 \geq 0$. For large $\tau_0 \gg 1$ these solutions go over to the asymptotic solutions of Sobolev. Thus, for the case of pure scattering for the resolvent function introduced by Sobolev we obtain the expression

$$\Phi(\tau, \tau_0) = \Phi(\tau) - C(\tau_0)\frac{\tau\alpha(\tau_0) + q(\tau) + q(\tau_0) - q(\tau_0 - \tau)}{\tau_0\alpha(\tau_0) + 2q(\tau_0)}, \quad (11)$$

where q is the Hopf function, $\alpha = 2 - C/\sqrt{3}$, and $C(\tau)$ is a function (see below) varying from ~ 2 at $\tau = 0$ to $\sqrt{3}$ at $\tau = \infty$. We give another equation

$$\varphi(\tau_0, \eta) = \varphi(\eta) - \frac{\tilde{F}(\tau_0, \eta)/\sqrt{3}}{\alpha\tau_0 + 2q(\tau_0)}[\varphi(\eta) + F(\tau_0, \eta) - \tilde{F}(\tau_0, \eta)] \quad (12)$$

for the Ambartsumyan function for a layer of thickness τ_0. Numerical results indicate that the error in Eq. (12) for all $\tau_0 \geq 0$ is less than or equal to one unit in the third digit, which corresponds to the accuracy of the Sobolev asymptotic solutions at $\tau_0 \approx 2$. For $\tau_0 \approx 2$ the accuracy of Eq. (12) is an order of magnitude greater.

Such approximate solutions to Eq. (3) are obtained if in expression (9) for the kernel $Z(\tau_0, \eta, \zeta)$ we replace the function $\tilde{F}(\tau_0, \zeta)$ by its asymptotic form for large τ_0: $\tilde{F}(\tau_0, \zeta) \approx \zeta C(\tau_0)/(1 + k\zeta)$ [in (11) and (12) $k = 0$, $\tilde{F}(\tau_0, \eta) \approx \eta C(\tau_0)$, and $C(\tau_0) \approx \tilde{F}(\tau_0, 1)$], retaining thereby the exact expression for $F(\tau_0, \eta)$. This is due to the physical circumstance that the quantity $\tilde{F}$ describes a photon moving deep into the medium, and in distinction to F, reaches the asymptotic domain at comparatively lesser depths. The Sobolev solutions, however, correspond to substituting the asymptotic forms of both these functions. For $\tau_0 \gg 1$ (11) and (12) go over to the asymptotic equations of Sobolev.[8] In a particular problem a similar result was found "intuitively" by Yamamoto.[9]

Analogous solutions can be found also for an aspherical scattering indicatrix. For an indicatrix greatly elongated forward these solutions can be highly effective, since the effect, in which a photon moving inward in a semiinfinite medium reaches the asymptotic domain, is intensified with increasing elongation of the indicatrix, while with increasing elongation of the indicatrix the asymptotic solutions of Sobolev retain their accuracy only when the thickness of the layer τ_0 is increased.

The examples given indicate the great effectiveness of the proposed method for studying the radiative transfer process in a layer of finite optical thickness. Preliminary work in this direction shows that an overwhelming majority of the results of radiative transfer theory can be obtained, avoiding the well-known mathematical complications, by proceeding from the physical properties of the operators Y and Z. Moreover, many useful relations obtained in this manner turn out to be completely new.

In addition, the development of this approach in application to semiinfinite media leads to an improved method for solving problems in a half-space. This method is distinguished by its generality and physical clarity, and in mathematical respects is almost elementary. In its essence, it is none other than the invariance method in the usual sense of Ambartsumyan. We given one example. It is clear that the solution to the Milne problem is given by the equation Gu = 0. In the simplest problem of a three-dimensional medium for G we have (8), and in terms of the intensities,

$$G\varphi = \frac{\varphi(\eta)}{\eta} - \frac{\lambda}{2}\frac{\varphi(\eta)}{\eta}\int_0^1 \varphi(\mu)\,d\mu = \frac{\varphi(\eta)}{\eta}\sqrt{1-\lambda}. \quad (13)$$

For pure scattering we obtain $G\varphi = 0$, i.e., $u(\eta) = \varphi(\eta)$ (ref. 2).

[1]V. A. Ambartsumyan, Dokl. Akad. Nauk SSSR, 38, No. 8, 257 (1943).
[2]V. A. Ambartsumyan, Scientific Works [in Russian], Vol. 1, Erevan (1960).
[3]V. V. Sobolev, A Treatise on Radiative Transfer, Van Nostrand, Princeton, New Jersey (1963).
[4]V. V. Sobolov, Light Scattering in Planetary Atmospheres [in Russian], Moscow (1972).
[5]N. B. Engibaryan and M. A. Mnatsakanyan, Dokl. Akad. Nauk SSSR, 217, 533 (1974) [Sov. Phys.–Dokl., 19, 391 (1975)].
[6]M. A. Mnatsakanyan, Soobshch. Byurakanskoi Astrofizich. Observ., 46, 93 (1975).
[7]É. Kh. Danielyan and M. A. Mnatsakanyan, Soobshch. Byurakanskoi Astrofizich. Observ., 46, 101 (1975).
[8]V. V. Sobolev, Dokl. Akad. Nauk SSSR, 155, 316 (1964) [Sov. Phys.–Dokl., 9, 222 (1964)].
[9]V. V. Sobolev, Astron. Zh., 39, 229 (1962) [Sov. Astron., 6, 178 (1962)].

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