ANALYTIC SOLUTIONS OF HIGH ACCURACY TO THE PROBLEM OF MONOCHROMATIC SCATTERING OF LIGHT IN A PLANE LAYER

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Approximate analytic solutions are found to the problem of the transfer of monochromatic radiation in a three-dimensional plane layer of finite optical thickness in the case of a spherical phase function, the approximate solutions being valid for an arbitrary value of the angular argument η on the complete complex plane. These solutions are asymptotic with respect to the thickness of the layer and have an exceptionally high degree of accuracy. In the worst case — small thicknesses and conservative scattering — the error does not exceed a few tenths of a percent. With increasing thickness of the layer and decreasing λ albedo of single scattering the accuracy of the solutions increases. In the limit $\eta \to \infty$, they yield highly accurate solutions to the problem of the mean number of scatterings of a photon in a layer. For large thicknesses of the layer, in the corresponding approximation, the solutions go over into the well-known asymptotic solutions for $\eta \leqslant 1$ and also $\eta > 1$, and also for the problem of the mean number of scatterings of a photon.

1. Introduction

Explicit solutions for the problem of monochromatic scattering of light in a plane-parallel layer are known only for the cases of homogeneous infinite or semi-infinite media [1-4]. The finding exact solutions for the more general problem of a layer of finite optical thickness would of course be an important step in the analytic theory of radiative transfer. Nevertheless, such solutions have not hitherto been found in a closed form.

However, even if exact analytic solutions to the problem of a layer of finite thickness were available, one would in practice need various approximations to them by comparatively simple expressions. Judging from the known exact solutions for the case of a semi-infinite layer, the corresponding problem is by no means the easiest.

From this point of view, it would undoubtedly be worthwhile obtaining directly approximate but rather accurate analytic solutions to the problem of monochromatic scattering of light in a layer of finite thickness. The condition of analyticity, i.e., validity of the solutions on the entire complex plane of values of the angular argument η, is valuable from the point of view of the analysis of such solutions. In addition, it is not only values of η in the interval $(-1, 1)$ that have direct physical meaning. Also of interest is the case when $\eta = 1/k$ is a root of the characteristic equation [1]

$$\Lambda(\eta) = 1 - \frac{\lambda}{2}\eta \ln\frac{1+\eta}{1-\eta}, \quad \Lambda\left(\frac{1}{k}\right) = 0, \tag{1}$$

and the limiting case $\eta \to \infty$. The latter case corresponds to the problem of the mean number of scatterings of a photon in the layer and is of independent interest.

In addition, such solutions for arbitrary value of η to the problem of monochromatic scattering are helpful in the derivation and understanding of more general solutions, for example, the problem of incoherent scattering, for which the values of the corresponding argument $z = \alpha(x)/\eta$ are physical on the complete infinite axis $(-\infty < z < \infty)$.

The present paper is devoted to the derivation and discussion of approximate analytic solutions to the problem of monochromatic scattering of light in a homogeneous layer of

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finite optical thickness. These solutions have an exceptionally high accuracy for all thicknesses of the layer and on the complete complex plane (η). In particular, in the limit $\eta \to \infty$ they yield highly accurate solutions to the problem of the mean number of photon scatterings in a layer of finite thickness.

The main result of the paper is that the solution to the arbitrary problem for a layer of finite thickness in terms of elementary expressions can be reduced explicitly (but approximately) to the known exact solutions of the corresponding problem for a semi-infinite layer.

In this respect, there is a similarity to the well-known asymptotic solutions [3-5] to the problem of light scattering in a layer of large optical thickness. These last solutions, however, have sufficient accuracy only from a definite thickness of the layer, this thickness increasing rapidly with decreasing albedo λ of simple scattering and with increasing elongation of the phase function. The solutions obtained below, which have the nature of asymptotic solutions with respect to the layer thickness, have a much higher degree of accuracy for any layer thickness. At large thicknesses, in the corresponding approximation, they yield in special cases asymptotic solutions to the problem of the emerging radiation [3-5] and the problem of the mean number of scatterings of a photon. A distinctive feature of the solutions is that their accuracy increases with decreasing λ.

The solutions considered below are actually analytic continuations of the solutions found earlier by the author [6], which are real for $\eta \leqslant 1$, to the complete complex plane of (η). We recall that the latter contain as a special result the "intuitive" solution of Yamamoto, which aroused great interest in the literature on account of its high accuracy and was long suspected of being exact [6].

Below, we restrict ourselves to considering only the case of isotropic scattering; a separate paper sill be devoted to anisotropic scattering.

2. Formulation of the Problem and Basic Equations

Suppose that in a layer of finite thickness τ_0 there is a photon traveling at depth τ in the direction ζ. The angular argument ζ is the cosine of the angle between the direction in which the photon is traveling and the normal to one of the boundaries of the layer, so that the value of $\zeta = 0$ corresponds to a photon absorbed at the given depth. We consider the more general case of an arbitrary distribution function $g(\tau, \zeta)$ of the "initial" photon over the depth τ and the direction ζ; this replaces the usual distribution function of the "initial sources" $g(\tau)$.

We denote by $j(\tau, \eta) \equiv j(\tau, \tau_0, \eta; g)$ the radiation field in the medium; it is the probability density that a photon is at some time at depth τ traveling in the direction η. By $j(\eta) \equiv j(0, \eta)$ we denote the angular distribution of the radiation emerging through the boundary of the layer. Below, we adopt the probability treatment of photon scattering processes in the medium.

We add to the considered layer of thickness τ_0 a semi-infinite medium (without sources) and consider the total semi-infinite medium with given distribution $g(\tau, \zeta)$ of the initial photons in the boundary layer of thickness τ_0. We denote by $J(\tau, \eta)$ the radiation field in the total semi-infinite medium. We also introduce the Green's function $\Gamma(\tau', \tau, \eta, \zeta)$ for the semi-infinite medium; it is the probability that a photon traveling at depth τ in the direction ζ will at some time be traveling at the depth τ' in the direction η.

Then it follows from simple physical considerations that [6]

$$J(\tau, \eta) = j(\tau, \eta) + \int_0^1 \Gamma(\tau, \tau_0, \eta, -\mu)\, j(\mu)\, d\mu. \qquad (2)$$

This relation makes it possible to reduce explicitly the solution to the problem of the interior radiation field $j(\tau, \eta)$ in a layer of finite thickness τ_0 to the solution of two special problems: the distribution of the radiation emerging through the boundary of this layer and the finding of the Green's function of a semi-infinite medium.

We now consider the determination of the angular distribution of the radiation which emerges through the boundary of the considered layer of thickness τ_0,

$$j^+(\eta) \equiv j(0, \eta), \quad j^-(\eta) \equiv j(\tau_0, -\eta)$$

for given distribution function $g(\tau, \zeta)$.

Adding to one or other boundary of the layer a semi-infinite medium and denoting by $J^+(\eta)$ or $J^-(\eta)$ the distribution of the radiation emerging through the boundary of the corresponding total semi-infinite medium, we obtain [6]

$$J^+(\eta) = j^+(\eta) + \int_{0}^{1} Z(\tau_0, \eta, \mu) j^-(\mu)\, d\mu, \quad J^-(\eta) = j^-(\eta) + \int_{0}^{1} Z(\tau_0, \eta, \mu) j^+(\mu)\, d\mu. \tag{3}$$

Here, $Z(\tau, \eta, \mu) = \Gamma(0, \tau, \eta, -\mu)$ is the probability of emergence from the semi-infinite medium of a photon traveling at depth τ in the direction μ within the medium.

The relations (3) establish a connection between the solution to the problem of the radiation emerging from a layer of finite thickness and the solution of the same problem for semi-infinite layer. We shall assume that the characteristics $J^{\pm}$ and Z of the semi-infinite medium are known and regard the relations (3) as equations for determining the characteristics $j^{\pm}(\eta)$ of the layer of thickness τ_0.

From (3) there follow the separate equations

$$S(\eta) = s(\eta) + \int_0^1 Z(\tau_0, \eta, \mu)\, s(\mu)\, d\mu, \tag{4}$$

$$H(\eta) = h(\eta) - \int_0^1 Z(\tau_0, \eta, \mu)\, h(\mu)\, d\mu, \tag{5}$$

for the sum and difference of the required quantities

$$s(\eta) \equiv j^+(\eta) + j^-(\eta), \quad h(\eta) \equiv j^+(\eta) - j^-(\eta), \tag{6}$$

where

$$S(\eta) \equiv J^+(\eta) + J^-(\eta), \quad H(\eta) \equiv J^+(\eta) - J^-(\eta). \tag{7}$$

Here and below, the upper case letters denote the quantities corresponding to the semi-infinite medium, and the corresponding lower case letters those relating to the layer of thickness τ_0.

Equations (4) and (5) are linear integral Fredholm equations of the second kind. It is noteworthy that in them τ, τ_0, and ζ, like the distribution $g(\tau, \zeta)$, serve as parameters. This circumstance makes direct solution of Eqs. (4) and (5) by the method of discretization with respect to the directions particularly effective. Moreover, irrespective of the actual formulation of the problem, for given layer thickness it is necessary to invert the same two matrices $(I + Z(\tau_0))_{ij}$ and $(I - Z(\tau_0))_{ij}$ in all cases.

It is obvious that relations analogous to (2)-(7) can also be written down in the more general cases of anisotropic scattering, arbitrary frequency redistribution law, when allowance is made for polarization, and so forth.

3. Approximate Analytic Solutions

We give the final expressions for the approximate analytic solutions of Eqs. (4) and (5) expressed in terms of the characteristics for the semi-infinite layer. These expressions are derived in Appendix II.

Let $P(\tau, \eta)$ be the probability of emergence in direction η of a photon absorbed at depth τ of the semi-infinite medium, and $P(0, \eta) = (\lambda/2)\varphi(\eta)$ be Ambartsumyan's function. Using them, we construct the "correction" functions

$$\alpha(\tau, \eta) = \frac{\lambda}{2} C(\tau) \frac{\eta\varphi(\eta)}{1 - k\eta}, \quad \beta(\tau, \eta) = P(\tau, \eta) - \alpha(\tau, \eta), \tag{8}$$

where k is the root of the characteristic equation (1), and $C(\tau)$ is a characteristic of the semi-infinite layer to be determined below (Sec. 5).

We introduce the following notation for the integral operators:

$$f_{\pm\eta} \equiv \int_0^1 \frac{f(\mu)\,d\mu}{\eta \pm \mu}, \qquad f_{\pm k} \equiv \int_0^1 \frac{f(\mu)\,d\mu}{1 \pm k\mu}. \tag{9}$$

We note that

$$f_{\pm k} = \lim_{\eta \to 1/k} (\eta f_{\pm\eta}). \tag{10}$$

The approximate solutions to the problem of radiation emerging through a boundary of the layer of thickness τ_0 can be expressed by means of the corrections α and β in the form

$$s(\eta) = S(\eta) - a(\tau_0, \eta)\, s_k - \eta\beta(\tau_0, \eta)\, s_\eta, \quad h(\eta) = H(\eta) + a(\tau_0, \eta)\, h_k + \eta\beta(\tau_0, \eta)\, h_\eta, \tag{11}$$

where

$$s_\eta = \left\{(S_\eta - a_\eta(\tau_0)\, s_k) - \eta\beta_\eta(\tau_0)\left[S_{-\eta} + \frac{2}{\lambda}\frac{\Lambda(\eta)}{\eta} a(\tau_0, \eta)\, s_k\right]\right\} / D(\tau_0, \eta),$$

$$h_\eta = \left\{(H_\eta + a_\eta(\tau_0)\, h_k) + \eta\beta_\eta(\tau_0)\left[H_{-\eta} - \frac{2}{\lambda}\frac{\Lambda(\eta)}{\eta} a(\tau_0, \eta)\, h_k\right]\right\} / D(\tau_0, \eta), \tag{12}$$

$$D(\tau, \eta) = 1 - \eta\beta_\eta(\tau)\left[e^{-\tau/\eta} - \frac{2}{\lambda}\Lambda(\eta)\,\beta(\tau, \eta)\right], \tag{13}$$

and s_k and h_k are determined from (12) by the limit (10) $\eta \to 1/k$:

$$s_k = (S_k - \beta_k(\tau_0)\, S_{-k})/d^+(\tau_0), \quad h_k = (H_k + \beta_k(\tau_0)\, H_{-k})/d^-(\tau_0), \tag{14}$$

where

$$d^\pm(\tau) = (1 - \beta_k(\tau)\, e^{-k\tau}) \pm a_k(\tau). \tag{15}$$

The solutions for the case of conservative scattering, $\lambda = 1$, follow from this in the limit $k \to 0$ (see Appendix II). They have the same form as (11)-(15) except that $f_{\pm k}$ must be replaced by

$$f_{\pm k} \to f_0 \equiv \int_0^1 f(\mu)\, d\mu,$$

and

$$s_0 = 1, \quad h_0 = \frac{G}{\alpha\tau_0 + 2q(\tau_0)}. \tag{16}$$

where

$$G = (\tau_0 - 2\tau)\,\alpha + 2\,[q(\tau_0 - \tau) - q(\tau)] + \frac{2}{\sqrt{3}}[F(\tau, \zeta) + \tilde{F}(\tau_0 - \tau, \zeta) - \zeta C(\tau_0)].$$

Here $\alpha = 2 - C(\tau_0)/\sqrt{3}$, $q(\tau)$ is Hopf's function, and the functions $F(\tau, \eta)$ and $\tilde{F}(\tau, \zeta)$ are also characteristics of the semi-infinite medium (see Sec. 5).

Thus, the required solutions for a layer of finite thickness can be expressed in an elementary (algebraic) manner in terms of S, H, α, β, $C(\tau_0)$, which characterize the semi-infinite medium, and integrals of the type (9). These last are ultimately expressed very simply in terms of the two functions $F(\tau, \eta)$ and $\tilde{F}(\tau, \zeta)$, which play a fundamental part in our approach to the solution of radiative transfer problems in a plane layer [7]. These expressions are given in Appendix I.

4. Special Problems and Numerical Results

For $g(\tau', \zeta') = \delta(\tau' - \tau)\,\delta(\zeta' - \zeta)$, our problem goes over into the problem of determining the Green's function for a layer of finite thickness and, in particular, the surface Green's function

$$r(0, \tau, \tau_0, \eta, \zeta) = \begin{cases} y(\tau, \tau_0, \eta, \zeta), & \zeta \geqslant 0, \\ z(\tau, \tau_0, \eta, \zeta), & \zeta \leqslant 0. \end{cases}$$

For $\tau = 0$ and $\tau = \tau_0$, we obtain from this the problem of reflection, $y(\tau_0, \tau_0, \eta, \zeta) = q(\tau_0, \eta, \zeta)$ and transmission, $z(0, \tau_0, \eta, \zeta) = r(\tau_0, \eta, \zeta)$, and for $\zeta = 0$ the problem of determining the probability of emergence of a photon absorbed at depth τ in the layer of thickness τ_0. The solution to this last problem follows from (11)-(16) if one makes the substitution $\zeta = 0$:

$$p(\tau, \tau_0, \eta) = P(\tau, \eta) - a(\tau_0, \eta) M(\tau, \tau_0) - \eta\beta(\tau_0, \eta)[P_\eta(\tau_0 - \tau) - a_\eta(\tau_0) M(\tau_0 - \tau, \tau_0)],$$

where

$$M(\tau, \tau_0) = \frac{(1 - \beta_k(\tau_0) e^{-k\tau_0}) m(\tau, \tau_0) - a_k(\tau_0) m(\tau_0 - \tau, \tau_0)}{(1 - \beta_k(\tau_0) e^{-k\tau_0})^2 - a_k^2(\tau_0)}, \quad m(\tau, \tau_0) = P_*(\tau_0 - \tau) - \beta_k(\tau_0) e^{-k\tau}. \tag{17}$$

Setting $\tau = 0$ and $\tau = \tau_0$ in these solutions, we obtain the following approximate expressions for Ambartsumyan's functions φ and ψ:

$$\begin{aligned} \frac{\lambda}{2}\varphi(\tau_0, \eta) &= \frac{\lambda}{2}\varphi(\eta) - \frac{s_k - h_k}{2} a(\tau_0, \eta) - \eta\beta(\tau_0, \eta)\left[P_\eta(\tau_0) - \frac{s_k + h_k}{2} a_\eta(\tau_0)\right], \\ \frac{\lambda}{2}\psi(\tau_0, \eta) &= P(\tau_0, \eta) - \frac{s_k + h_k}{2} a(\tau_0, \eta) - \eta\beta(\tau_0, \eta)\left[\frac{\lambda}{2}\varphi_\eta - \frac{s_k - h_k}{2} a_\eta(\tau_0)\right], \end{aligned} \tag{18}$$

where

$$s_k = 1 - \frac{1}{d^+(\tau_0)\varphi(1/k)}, \qquad h_k = 1 - \frac{1}{d^-(\tau_0)\varphi(1/k)}.$$

In the case of conservative scattering, we set $k = 0$ in (18),

$$s_0 = 1, \quad h_0 = \pm\left(\frac{2/\sqrt{3}}{\alpha\tau_0 + 2q(\tau_0)}\right), \tag{19}$$

the plus sign corresponding to the expression for φ, the minus sign to that for ψ.

In Tables 1 and 2 we give the functions $\varphi_\lambda(\tau_0, 1)$ and $\psi_\lambda(\tau_0, 1)$ for a number of values of τ_0 and λ calculated in accordance with the approximate expressions (18) and (19). For comparison, the exact values, taken from [8], are also given in the tables.

We see that the greatest error of the approximate solutions is attained at $\lambda = 1$ and small thicknesses τ_0 and is ~0.3-0.4%. With increasing τ_0 and decreasing λ the accuracy of the approximate solutions increases strongly. For example, for $\tau_0 = 2$ and $\lambda = 0.5$ the relative error is 0.002% and 0.02% for the φ and ψ functions, respectively, whereas the error of the known asymptotic solutions (see Sec. 7) in this case is ~1% for the φ function and more than 200% for the ψ function.

The functions $\varphi(\tau_0, \eta)$ and $\psi(\tau_0, \eta)$ have the same error at all values of η. In particular, in the limit $\eta \to \infty$ (mean number of scatterings) in the case of conservative scattering we obtain from (18)

TABLE 1. Approximate and Exact Values of $\varphi_\lambda(\tau_0, 1)$

τ_0 \ λ	1	0.99	0.95	0.9	0.5	
0	1.0002 1.0000	1.0001 1.0000	0.9999 1.0000	1.0000 1.0000	1.0000 1.0000	approx. exact
0.1	1.162 ?	1.160 ?	1.152 ?	1.143 ?	1.073 ?	approx. exact
0.2	1.268 1.265	1.265 1.261	1.250 1.247	1.233 1.230	1.114 1.113	approx. exact
0.4	1.432 1.429	1.425 1.422	1.398 1.395	1.366 1.363	1.1645 1.1639	approx. exact
1	1.759 1.757	1.740 1.739	1.673 1.672	1.5980 1.5972	1.2263 1.2262	approx. exact
2	2.0708 2.0707	2.0313 2.0311	1.8948 1.8947	1.7607 1.7606	1.24776 1.24774	approx. exact

TABLE 2. Approximate and Exact Values of $\psi_\lambda(\tau_0, 1)$

τ_0 \ λ	1	0.99	0.95	0.9	0.5	
0	0.9998 1.0000	0.9999 1.0000	1.0001 1.0000	1.0000 1.0000	1.0000 1.0000	approx. exact
0.1	1.065 ?	1.063 ?	1.055 ?	1.046 ?	0.977 ?	approx. exact
0.2	1.078 1.074	1.075 1.071	1.061 1.057	1.044 1.041	0.928 0.927	approx. exact
0.4	1.070 1.066	1.063 1.059	1.037 1.034	1.007 1.004	0.8193 0.8187	approx. exact
1	0.968 0.966	0.952 0.950	0.8926 0.8913	0.8283 0.8273	0.5276 0.5274	approx. exact
2	0.7810 0.7808	0.7494 0.7494	0.6431 0.6428	0.5439 0.5437	0.23324 0.23318	approx. exact

$$\varphi(\tau_0, \infty) = \psi(\tau_0, \infty) = \frac{\sqrt{3}}{2}[a\tau_0 + 2q(\tau_0)], \quad \lambda = 1. \tag{20}$$

In Table 3, we give the numerical values calculated in accordance with (20), the exact values taken from [9], and, for comparison, the values given by the asymptotic formula of [4, 10] corresponding to large τ_0 in (20):

$$\varphi(\tau_0, \infty) = \psi(\tau_0, \infty) = \frac{\sqrt{3}}{2}(\tau_0 + 2q(\infty)).$$

In Table 3, we also give the corresponding relative errors Δ in (%).

5. Basic Approximation

Every approximate solution to the problem (4), (5) corresponds to some approximation of the kernel Z of these equations. For example, the asymptotic solutions of [3-5] correspond to the approximation

$$Z(\tau, \eta, \zeta) \approx \frac{Ae^{-k\tau}\eta}{(1-k\eta)(1+k\zeta)}.$$

The kernel Z can be expressed in terms of the two functions F and $\tilde{F}$ in accordance with (A.I.1). These last are defined as

$$F(\tau, \eta) = \eta\int_0^1 Y(\tau, \mu, \eta)\,d\mu/\mu, \qquad \tilde{F}(\tau, \eta) = \eta\int_0^1 Z(\tau, \mu, \eta)\,d\mu/\mu, \tag{21}$$

and they admit the following physical interpretation. A photon absorbed at the boundary of the semi-infinite layer produces at depth τ a radiation field described by the function $F(\tau, \eta)$ for $\eta \leqslant 0$ and $\tilde{F}(\tau, \eta)$ for $\eta \geqslant 0$. At large τ, the variables separate in the function $\tilde{F}(\tau, \eta)$: $\tilde{F}(\tau, \eta) \approx Ae^{-k\tau}\eta/(1 + k\eta)$. In [6], on the basis of the physical meaning of $\tilde{F}$, it was established that the variables separate in it with fairly high accuracy for

TABLE 3. Exact, Approximate, and Asymptotic Values of $\varphi_1(\tau_0, \infty)$

τ_0	Exact	Approx.	Δ %	Asympt.	Δ %
0	1.000	1.000	0	1.23	23
0.1	?	1.169	?	1.32	13
0.2	1.287	1.291	0.3	1.40	8
1	2.067	2.069	0.1	2.10	1.4

all $\tau \geqslant 0$, so that

$$\tilde{F}(\tau, \eta) \approx C(\tau) \frac{\eta}{1+k\eta}, \tag{22}$$

whereas the variables separate in the function $F(\tau, \eta)$ with the same accuracy only at comparatively large τ.

The approximation (22) is true in general for small η, say $\eta \lesssim 1$. It does not hold at large η. Therefore, to obtain highly accurate solutions of the problem (4), (5) one should use the approximation (22) only on the interval $\eta \in (0, 1)$. This is possible, since the integration in Eqs. (4) and (5) is over this interval.

In the solution of Eqs. (4) and (5) described in Appendix II we retain in the expression for $Z(\tau_0, \eta, \mu)$ the exact function $F(\tau_0, \eta)$, and for $\tilde{F}(\tau_0, \mu)$ we use the approximation (22), but only on the interval $\mu \in (0, 1)$. In other words, the solutions (11)-(16) are based on the unique approximation

$$\int_0^1 \tilde{F}(\tau_0, \mu) f(\mu)\, d\mu \approx C(\tau_0) \int_0^1 \frac{\mu}{1+k\mu} f(\mu)\, d\mu, \tag{23}$$

where τ_0 is the thickness of the layer and the part of $f(\mu)$ is played by the functions $s(\mu)/(\eta \pm \mu)$ and $h(\mu)/(\eta \pm \mu)$.

The integral on the left-hand side of (33) does not have singularities except for the case $\eta = 1$, when we are dealing with functions of the type $f(\mu) \sim 1/(1 - \mu)$. In this case, it diverges at the upper limit, and to ensure the same behavior of both sides of the approximation (23) in the neighborhood of $\eta = 1$, we must impose on the function $C(\tau_0)$ (which is to a certain degree arbitrary) the requirement

$$C(\tau_0) = (1+k)\tilde{F}(\tau_0, 1). \tag{24}$$

With this choice of $C(\tau_0)$ the approximation (23) obviously ensures a fairly high accuracy of the solutions for all values of η.

The approximation (23) is satisfied better, the greater is the thickness τ_0. Therefore, the above solutions have an asymptotic nature — their accuracy increases with increasing τ_0. Another property of the approximation (23) is the increase in its accuracy with decreasing value of λ, the albedo of single scattering. This explains the increase in the accuracy of our solutions with decreasing λ noted in Sec. 4. Thus, the case of small thicknesses and conservative scattering is the best in the sense of the accuracy of the solutions considered in the paper.

In Fig. 1, we have plotted the graph of the function $\tilde{F}_\lambda(\tau_0, \eta)$ for the case of conservative scattering, $\lambda = 1$, at the values $\tau_0 = 0$ and $\tau_0 = \infty$ on the interval $\eta \in (0, 1)$. The approximation $\tilde{F}(\tau_0, \eta) \approx C(\tau_0)\eta$ corresponding to (22) is shown by the straight line 1. For $\tau_0 = \infty$, there is exact coincidence: $\tilde{F}(\infty, \eta) = \sqrt{3}\eta$.

The approximate solutions found in Sec. 3 can be improved. The improvement is based on a more accurate approximation of the function $\tilde{F}(\tau, \eta)$ on the interval $(0 < \eta < 1)$. For example, it can be represented in the form

$$\tilde{F}(\tau_0, \eta) = \frac{c_1(\tau_0)\eta - c_2(\tau_0)\eta^2}{1+k\eta}. \tag{25}$$

For $\lambda = 1$, this approximation is shown in Fig. 1 by the broken curve 3; for it,

$$c_1(\tau_0) = 2\tilde{F}(\tau_0, 1) - \tilde{F}'_\eta(\tau_0, 1), \quad c_2(\tau_0) = \tilde{F}'_\eta(\tau_0, 1) - \tilde{F}(\tau_0, 1), \tag{26}$$

In this case, Eqs. (4) and (5) also admit solution in a closed form, this having exceptionally high accuracy. We do not give this solution here because it is cumbersome.

In Fig. 1 we have also plotted the straight line 2, which represents the different approximation $\tilde{F}(\tau_0, \eta) \approx (c(\tau_0)\eta + d(\tau_0))/(1 + k\eta)$ for $\lambda = 1$: $c = c_1 - 2c_2$, $d = c_2$. The two approximations, i.e., the one used in the paper (the straight line 1) and the one given by the straight line 2, bound the exact behavior of the function $F(\tau_0, \eta)$ for $0 < \eta < 1$ above and below. Combining the solutions corresponding to these two approximations, we can obtain for the solution of the problem (4), (5) upper and lower bounds, and from them estimate the error of our solutions. Analysis shows that this error certainly does not

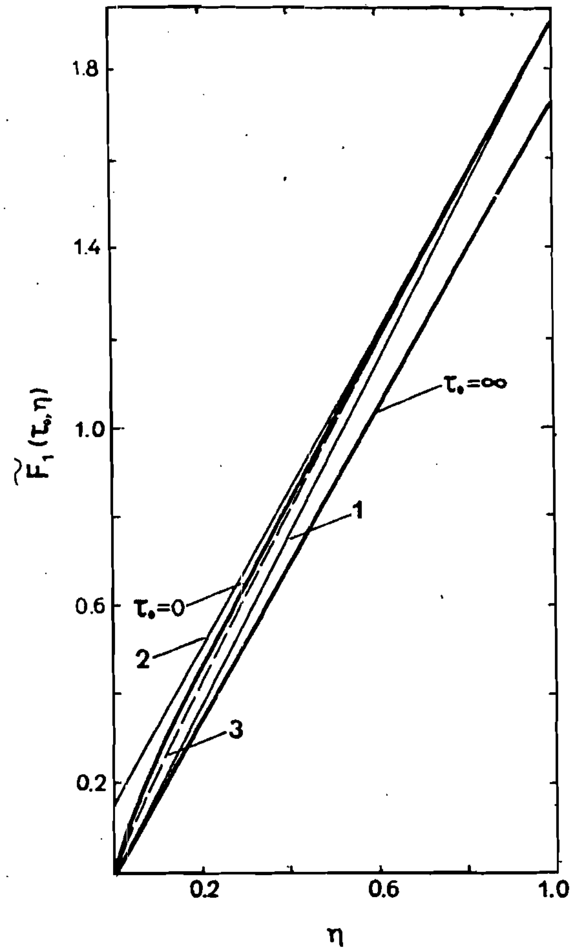

Fig. 1. $\tilde{F}_\lambda(\tau_0, \eta)$ in the case of conservative scattering for $\tau_0 = 0$ and $\tau_0 = \infty$ and various approximations of this function on the interval $0 \leqslant \eta \leqslant 1$.

exceed 1% for all values of τ_0, η, and λ.

It should be noted however that the error of the approximate solutions, because of their asymptotic nature, depends on the actual formulation of the problem, i.e., the distribution $g(\tau, \zeta)$. The greatest error in the problem of the emerging radiation corresponds clearly to the case when the sources are concentrated at the far boundary of the layer, as, for example, in the case of the solution of the ψ function problem. It is also clear that this difference must be small in the case of small thicknesses of the layer.

6. Solution at Small η

For small values of the angular argument $\eta \leqslant 1$, we can use the approximation (22) explicitly, i.e., in the solutions (11)-(16) we can ignore

$$\beta_\eta(\tau_0) = \frac{1}{\varphi(\eta)}\left[\frac{\bar{F}(\tau_0, \eta)}{\eta} - \frac{C(\tau_0)}{1+k\eta}\right] \to 0. \tag{27}$$

Then we obtain the solutions found in [6]. They have the form of (11) and (14), where s_η and h_η are now determined from

$$s_\eta = S_\eta - a_\eta(\tau_0)\, s_k, \quad h_\eta = H_\eta + a_\eta(\tau_0)\, h_k. \tag{28}$$

These solutions are valid for all λ. In particular, in the problem of the source function for conservative scattering, $\lambda = 1$, they are identical to the "intuitive" solution of Yamamoto [6].

Note that for $\eta \leqslant 1$ at layer thicknesses $\tau_0 \gtrsim 1$ the terms containing β_η become smaller than the error of the solutions themselves and can be ignored. In this respect, retention of the terms with β_η for $\tau_0 \lesssim 1$ leads to solutions that are more accurate than those found in [6].

If the value of λ is not very close to 1, comparatively simple expressions for the approximate solutions are given in [6]. They follow from substitution of not only (27) but also

$$\beta_k(\tau_0) = \frac{1}{\varphi(1/k)}\left[\bar{F}(\tau_0,\ 1/k) - \frac{C(\tau_0)}{2k}\right] \to 0 \tag{29}$$

and they have the form of (11), in which s_η and h_η are determined by (28), and s_k and h_k from

$$s_k = \frac{S_k}{1 + a_k(\tau_0)}, \quad h_k = \frac{H_k}{1 - a_k(\tau_0)}. \tag{30}$$

The error of these solutions is smaller, the smaller is the value of λ. This follows from the fact that as $\lambda \to 0$ we have $k \to 1$ and β_k tends to $[F(\tau_0,\ 1) - C(\tau_0)/2] \to 0$, which agrees exactly with the definition (24) for $C(\tau_0)$ as $k \to 1$. Actually, the definition (24) for $C(\tau_0)$ in [6] was found by other arguments. The expression (16) for the case of conservative scattering was found by a different way in [6].

It is clear that the solutions (11)-(16) can be obtained by using the derivation of the "quasiasymptotic" solutions of [6], eliminating at the appropriate places the inaccuracies introduced by the approximation (22) in explicit form, this approximation being retained only in integrands. This route is however less systematic and is inferior to the derivation given in Appendix II with regard to clarity and brevity.

7. Asymptotic Expressions

At large layer thicknesses, we can deduce from (11)-(16) comparatively simple expressions, sacrificing, of course, the accuracy of the solutions. The cases of small and large values of the angular argument η must be considered separately.

For $\tau_0 \gg 1$,

$$C(\tau_0) \to Ae^{-k\tau_0}, \qquad A = \left[\frac{\lambda}{2}\int_0^1 \frac{\mu\varphi(\mu)}{(1-k\mu)^2}\,d\mu\right]^{-1}, \quad a(\tau,\ \eta) = \frac{\lambda}{2}\,\frac{Ae^{-k\tau}\eta\varphi(\eta)}{1-k\eta}. \tag{31}$$

This corresponds to neglecting in the solutions (11)-(15) the quantities

$$\beta_k(\tau_0) \to 0, \quad \beta_\eta(\tau_0) \to 0. \tag{32}$$

For $\eta \leqslant 1$, we set

$$\beta_k(\tau_0,\ \eta) = P(\tau_0,\ \eta) - a(\tau_0,\ \eta) \to 0. \tag{33}$$

Then we obtain the general form of the asymptotic solutions:

$$s(\eta) = S(\eta) - a(\tau_0,\ \eta)\,s_k, \quad h(\eta) = H(\eta) + a(\tau_0,\ \eta)\,h_k, \tag{34}$$

where

$$s_k = \frac{S_k}{1 + a_k(\tau_0)}, \quad h_k = \frac{H_k}{1 - a_k(\tau_0)}, \quad a_k(\tau_0) = \frac{Ae^{-k\tau_0}}{2k\varphi(1/k)}. \tag{35}$$

For conservative scattering $\lambda = 1$, $k \to 0$

$$f_k \to f_0,\ s_0 = 1,\ h_0 = 1 - 2\,\frac{\tau + q(\tau) - q(\tau_0 - \tau) + q(\infty)}{\tau_0 + 2q(\infty)}. \tag{36}$$

The asymptotic solutions for the interior radiation field follow from (2):

$$j(\tau,\ \eta) \approx J(\tau,\ \eta) - P^*(\tau,\ \tau_0,\ \eta)\,j_k, \tag{37}$$

where

$$j_k = J_k(0)/[1 + P_k^*(0,\ \tau_0)]. \tag{38}$$

Here, $P^*(\tau, \tau_0, \eta) = J_M(\tau, \eta) e^{-k\tau_0}$ is the interior solution in the Milne problem [11].

The solutions (34)-(38) are also valid for the case of anisotropic scattering if in them we make the substitution

$$f_k \to \int_0^1 \frac{f(\mu)\, b(-\mu)}{1+k\mu}\, d\mu \tag{39}$$

where b(μ) determines the "deep regime" [1, 3]. Then they will correspond to the quantities averaged over the azimuth. For the higher harmonics, it follows in general from the relation (1), written down with allowance for the azimuthal dependence, that $j^m_{(\tau,\eta)} = J^m_{(\tau,\eta)}$, $m \geqslant 1$.

Note that the asymptotic solutions (37) hold for all $\tau \leqslant \tau_0$ (under the condition $\tau_0/2 \gg 1$), since $j(\eta)$ in them can be understood as either $j^+(\eta)$ or $j^-(\eta)$ from the problem (3).

An interesting conclusion can be drawn from the solutions given above: if the initial photons are concentrated near the boundaries of the layer, then in the case of conservative scattering the total angular distribution of the radiation emerging through both boundaries of the layer does not depend in the asymptotic approximation on the thickness τ_0 of the layer.

On the transition to the asymptotic expressions in the case of large values $\eta > 1$ of the angular argument, we must again use the approximations (32) but retain $\beta(\tau_0, \eta)$ in the approximation corresponding to the asymptotic behavior (4.40) in Ch. 8 of [4]:

$$\beta(\tau_0, \eta) \to \beta_{as}(\tau_0, \eta) = \frac{\lambda}{2} \frac{e^{-\tau_0/\eta}}{\Lambda(\eta)}. \tag{38}$$

Note that the denominators in (12) can be rewritten in the form

$$D(\tau_0, \eta) \approx 1 - \frac{2}{\lambda} \Lambda(\eta)\, \beta_\eta(\tau_0) \left[\beta_{as}(\tau_0, \eta) - \beta(\tau_0, \eta)\right].$$

The asymptotic behaviors now have the general form

$$s(\eta) = S(\eta) - a(\tau_0, \eta)\, s_k - \eta\beta_{as}(\tau_0, \eta)\, s_\eta, \qquad h(\eta) = H(\eta) + a(\tau_0, \eta)\, h_k + \eta\beta_{as}(\tau_0, \eta)\, h_\eta, \tag{39}$$

where s_k and h_k are determined from (35), and s_η and h_η from (28). For $\lambda = 1$, we again use the expressions (36).

In particular, setting $\zeta = 0$ in (39), we find an asymptotic expression for the probability of emergence of a photon absorbed at depth τ:

$$\begin{aligned} p(\tau, \tau_0, \eta) = P(\tau, \eta) - \lambda k a_k(\tau_0) \frac{\eta\varphi(\eta)}{1-k\eta} \frac{\bar{F}(\tau_0-\tau, 1/k) - a_k(\tau_0)\bar{F}(\tau, 1/k)}{1-a_k^2(\tau_0)} - \\ \frac{\lambda}{2} \frac{e^{-\tau_0/\eta}}{\Lambda(\eta)\varphi(\eta)} \left[\bar{F}(\tau_0-\tau, \eta) - 2k a_k(\tau_0) \frac{\eta}{1+k\eta} \frac{\bar{F}(\tau_0, 1/k) - a_k(\tau_0)\bar{F}(\tau_0-\tau, 1/k)}{1-a_k^2(\tau_0)}\right]. \end{aligned} \tag{40}$$

In the case of conservative scattering, this solution has the form

$$p(\tau, \tau_0, \eta) = P(\tau, \eta) + \frac{h_0-1}{\eta} a(\tau_0, \eta) + \beta_{as}(\tau_0, \eta) \frac{\eta}{\varphi(\eta)} \left[\sqrt{3}\frac{h_0+1}{2} - \frac{\bar{F}(\tau_0-\tau, \eta)}{\eta}\right]. \tag{41}$$

In the special cases $\tau = 0$ and $\tau = \tau_0$ we obtain from (40) and (41) the well-known asymptotic solutions (for $\eta > 1$) for the functions $\varphi(\tau_0, \eta)$ and $\psi(\tau_0, \eta)$ found by Ivanov ([4], Ch. 8, Eqs. (4.41)-(4.44)).

Appendix I

Auxiliary Expressions. In accordance with (6) and (7), the expressions of Sec. 3 give an approximate analytic solution to the problem of finding the radiation $j^{\pm}(\eta)$ emerging through the boundaries of a layer of thickness τ_0 for an arbitrary distribution $g(\tau, \zeta)$ of the initial photons. They are expressed in terms of the solutions $J^{\pm}(\eta)$ to the problem of the radiation which emerges from a semi-infinite medium containing initial photons in a boundary layer of thickness τ_0 distributed in accordance with the same law $g(\tau, \zeta)$ or $g(\tau_0 - \tau, -\zeta)$. After the determination of $j^{\pm}(\eta)$, the problem of the interior radiation field in the layer of thickness τ_0 reduces to calculation of the integral (2).

For a monodirectional source, when $g(\tau', \zeta') = \delta(\tau'-\tau)\,\delta(\zeta'-\zeta)$, the surface Green's function of a semi-infinite layer can serve as $J^{\pm}(\eta)$:

$$\Gamma(0, \tau, \eta, \zeta) \equiv X(\tau, \eta, \zeta) = \begin{cases} Y(\tau, \eta, \zeta), & \zeta \geqslant 0, \\ Z(\tau, \eta, \zeta), & \zeta \leqslant 0, \end{cases}$$

it being expressed by the simple equations

$$Y(\tau, \eta, \zeta) = \frac{\lambda}{2}\eta\varphi(\eta)\frac{F(\tau, \eta) - F(\tau, \zeta)}{\eta - \zeta} + e^{-\tau/\zeta}\delta(\eta - \zeta). \quad Z(\tau, \eta, \zeta) = \frac{\lambda}{2}\eta\varphi(\eta)\frac{F(\tau, \eta) + \tilde{F}(\tau, \zeta)}{\eta + \zeta}, \qquad (I.1)$$

in terms of the two functions $F(\tau, \eta)$ and $\tilde{F}(\tau, \zeta)$ of two arguments defined in Sec. 5. Ultimately, the solution to the problem (4), (5) can be expressed algebraically in terms of these two functions.

In the case of arbitrary function $g(\tau, \zeta)$, the quantities $J^{\pm}(\eta)$ are found by integrating the Green's function over the distribution g:

$$J^{+}(\eta) = \int_0^{\tau_0}\int_{-1}^{1} X(\tau', \eta, \mu')\, g(\tau', \mu')\, d\tau' d\mu', \quad J^{-}(\eta) = \int_0^{\tau_0}\int_{-1}^{1} X(\tau', \eta, \mu')\, g(\tau_0 - \tau', -\mu')\, d\tau' d\mu'.$$

An important circumstance here is that the operations of integration over the distribution g are performed on the characteristics of the semi-infinite layer, and not of the layer of finite thickness. Because of this, analytic integration is possible in a number of cases; numerical integration over the distribution g of the characteristics of a layer of finite thickness would lead to additional errors.

We now list the auxiliary formulas that express the characteristics of the semi-infinite medium (for isotropic scattering) in terms of the functions F and $\tilde{F}$.

Integrals of the type (9):

$$Y_\eta(\tau, \zeta) = \frac{\tilde{F}(\tau, \eta) + F(\tau, \zeta)}{(\eta + \zeta)\varphi(\eta)}, \qquad Y_k(\tau, \zeta) = \frac{\tilde{F}\left(\tau, \frac{1}{k}\right) + F(\tau, \zeta)}{(1 + k\zeta)\varphi(1/k)},$$

$$Z_\eta(\tau, \zeta) = \frac{\tilde{F}(\tau, \eta) - \tilde{F}(\tau, \zeta)}{(\eta - \zeta)\varphi(\eta)}, \qquad Z_k(\tau, \zeta) = \frac{\tilde{F}\left(\tau, \frac{1}{k}\right) - \tilde{F}(\tau, \zeta)}{(1 - k\zeta)\varphi(1/k)}, \qquad (I.2)$$

$$Y_{-\eta}(\tau, \zeta) = \frac{e^{-\tau/\eta}}{\eta - \zeta} - \frac{2}{\lambda}\frac{\Lambda(\eta)}{\eta}\left[Y(\tau, \eta, \zeta) - e^{-\tau/\zeta}\delta(\eta - \zeta)\right],$$

$$Y_{-k}(\tau, \zeta) = \frac{e^{-k\tau}}{1 - k\zeta}, \qquad Z_{-\eta}(\tau, \zeta) = \frac{e^{-\tau/\eta}}{\eta + \zeta} - \frac{2}{\lambda}\frac{\Lambda(\eta)}{\eta} Z(\tau, \eta, \zeta), \qquad Z_{-k}(\tau, \zeta) = \frac{e^{-k\tau}}{1 + k\zeta}.$$

For $\zeta = 0$:

$$Y(\tau, \eta, 0) = Z(\tau, \eta, 0) = P(\tau, \eta) = \frac{\lambda}{2}\varphi(\eta) F(\tau, \eta),$$

$$P_\eta(\tau) = \frac{\tilde{F}(\tau, \eta)}{\eta\varphi(\eta)}, \quad P_k(\tau) = \frac{\tilde{F}(\tau, 1/k)}{\varphi(1/k)}, \quad P_{-\eta}(\tau) = \frac{e^{-\tau/\eta}}{\eta} - \frac{2}{\lambda}\frac{\Lambda(\eta)}{\eta}P(\tau, \eta), \quad P_{-k}(\tau) = e^{-k\tau}. \qquad (I.3)$$

In particular, for $\tau = 0$:

$$\frac{\lambda}{2}\eta\varphi_\eta = 1 - \frac{1}{\varphi(\eta)}, \quad \frac{\lambda}{2}\varphi_k = 1 - \frac{1}{\varphi(1/k)}, \quad \frac{\lambda}{2}\eta\varphi_{-\eta} = 1 - \Lambda(\eta)\varphi(\eta), \quad \frac{\lambda}{2}\varphi_{-k} = 1. \qquad (I.4)$$

Integrals of $\alpha(\tau, \eta)$:

$$a_\eta(\eta) = \frac{C(\tau)}{(1 + k\eta)\varphi(\eta)}, \qquad a_k(\tau) = \frac{C(\tau)}{2k\varphi(1/k)}. \qquad (I.5)$$

The functions $F(\tau, \eta)$ and $\tilde{F}(\tau, \zeta)$ are tabulated in [12]. In practice, the simplest method of calculating these functions is by means of the recursion relation

$$F(\tau+t, \eta)=F(\tau, \eta)e^{-t/\eta}+\eta\int_0^1\left[\frac{\lambda}{2}\varphi(\mu)\frac{F(t, \mu)-F(t, \eta)}{\mu-\eta}\right]F(\tau, \mu)\,d\mu \tag{I.6}$$

and the following expression for $\tilde{F}$ in terms of F:

$$\tilde{F}(\tau, \zeta)=\frac{\lambda}{2}\zeta\varphi(\zeta)\int_0^1\frac{F(\tau, \mu)}{\eta+\mu}\varphi(\mu)\,d\mu. \tag{I.7}$$

For a small step t, the function $F(t, \eta)$ can be calculated by means of the resolvent function $\Phi(\tau)$ [2] (see, for example, [12]):

$$F(t, \eta)=e^{-t/\eta}+\int_0^t\Phi(\tau)e^{-\frac{t-\tau}{\eta}}\,d\tau. \tag{I.8}$$

We also give special values of the functions $F(\tau, \eta)$ and $\tilde{F}(\tau, \zeta)$:

$$F(0, \eta)=1, \quad \tilde{F}(0, \zeta)=\varphi(\zeta)-1, \quad \Phi(\tau)=\lim_{\eta\to 0}\frac{F(\tau, \eta)}{\eta}=\lim_{\zeta\to 0}\frac{\tilde{F}(\tau, \zeta)}{\zeta}, \tag{I.9}$$

and we also note that

$$\frac{1}{\eta}F(\tau, \eta)=\lim_{\mu\to 0}\frac{Y(\tau, \mu, \eta)}{\mu}, \quad \frac{1}{\zeta}\tilde{F}(\tau, \zeta)=\lim_{\mu\to 0}\frac{Z(\tau, \mu, \zeta)}{\mu}. \tag{I.10}$$

Appendix II

Solution of the Equations. The relations (4) and (5) are equations for the unknown functions $s(\eta)$ and $h(\eta)$ only for $\eta \leqslant 1$. For other values of η outside the interval (0, 1), they are explicit expressions for the unknown functions s and h in terms of their values for $\eta \leqslant 1$. Thus, if exact analytic solutions $s(\eta)$ and $h(\eta)$ for $\eta \in (0, 1)$ were known, the relations (4) and (5) would permit their analytic continuation to the complete complex plane of η. Since it is not possible to find an exact solution for s and h when $\eta \leqslant 1$, the question of their analytic continuation becomes all the harder. In general, approximate solutions valid for $\eta \leqslant 1$ cannot be extended to other values of η. For example, the approximate solutions found in [6] cannot be continued analytically, since one cannot ensure the same degree of error for all values of η. For validity of the approximate solutions on the complete complex plane of η, it is necessary that the approximation itself to which the problem is solved be satisfied with the given accuracy for all values of η. The approximation (23) used in the present paper has this property, but (22) does not.

We introduce the integral operators

$$\tilde{f}_{\mp\eta}\equiv\int_0^1\frac{\tilde{F}(\tau, \mu)}{\eta\pm\mu}f(\mu)\,d\mu, \qquad \tilde{f}_{\pm k}\equiv\int_0^1\frac{\tilde{F}(\tau, \mu)}{1\pm k\mu}f(\mu)\,d\mu. \tag{II.1}$$

Applying them to Eq. (4), we find

$$\begin{gathered}S_\eta=s_\eta+\eta P_\eta(\tau_0)s_{-\eta}-\tilde{s}_{-\eta}/\varphi(\eta), \quad S_k=s_k+P_k(\tau_0)s_{-k}-\tilde{s}_{-k}/\varphi(1/k),\\ S_{-\eta}=s_{-\eta}+e^{-\tau_0/\eta}s_\eta-\frac{2}{\lambda}\frac{\Lambda(\eta)}{\eta}[S(\eta)-s(\eta)], \quad S_{-k}=s_{-k}+e^{-k\tau_0}s_k.\end{gathered} \tag{II.2}$$

Similarly, from Eq. (5)

$$\begin{gathered}H_\eta=h_\eta-\eta P_\eta(\tau_0)h_{-\eta}+\tilde{h}_{-\eta}/\varphi(\eta), \quad H_k=h_k-P_k(\tau_0)h_{-k}+\tilde{h}_{-k}/\varphi(1/k),\\ H_{-\eta}=h_{-\eta}-e^{-\tau_0/\eta}h_\eta+\frac{2}{\lambda}\frac{\Lambda(\eta)}{\eta}[h(\eta)-H(\eta)], \quad H_{-k}=h_{-k}-e^{-k\tau_0}h_k.\end{gathered} \tag{II.3}$$

The original equations (4.5) in the notation (II.1) have the form

$$S(\eta)=s(\eta)+\eta P(\tau_0, \eta)s_\eta+\frac{\lambda}{2}\eta\varphi(\eta)\tilde{s}_\eta, \quad H(\eta)=h(\eta)-\eta P(\tau_0, \eta)h_\eta-\frac{\lambda}{2}\eta\varphi(\eta)\tilde{h}_\eta. \tag{II.4}$$

In all the equations, we express $s_{\pm\eta}$ and $h_{\pm\eta}$ in terms of $\tilde{s}_{\pm\eta}$ and $\tilde{h}_{\pm\eta}$, and to calculate the latter we use the approximation (23). Then we obtain

$$S_\eta = s_\eta + \eta\beta_\eta(\tau_0)s_{-\eta} + a_\eta(\tau_0)s_k, \quad S_k = s_k + \beta_k(\tau_0)s_{-k} + a_k(\tau_0)s_k,$$
$$S_{-\eta} = s_{-\eta} + e^{-\tau_0/\eta}s_\eta - \frac{2}{\lambda}\frac{\Lambda(\eta)}{\eta}[S(\eta) - s(\eta)], \quad S_{-k} = s_{-k} + e^{-k\tau_0}s_k, \tag{II.5}$$

and, similarly,

$$H_\eta = h_\eta - \eta\beta_\eta(\tau_0)h_{-\eta} - a_\eta(\tau_0)h_k, \quad H_k = h_k - \beta_k(\tau_0)h_{-k} - a_k(\tau_0)h_k,$$
$$H_{-\eta} = h_{-\eta} - e^{-\tau_0/\eta}h_\eta + \frac{2}{\lambda}\frac{\Lambda(\eta)}{\eta}[h(\eta) - H(\eta)], \; H_{-k} = h_{-k} - e^{-k\tau_0}h_k. \tag{II.6}$$

The solution of the system of algebraic equations (II.4)-(II.6) is given by the expressions (11)-(15) of Sec. 3.

To go over from these solutions to the solutions for the case of conservative scattering, we must go to the limit $\lambda \to 1$. Then in the expression for s_k it is sufficient to use expansions to the first power of k, while in the expression for h_k it is also necessary to retain the terms containing k^2. Note that in the limit $k \to 0$ an uncertainty arises only in the expression for h_k.

To derive Eq. (16), we obtained an expansion in small k for the resolvent function:

$$\Phi_\lambda(\tau) = \Phi_1(\tau) + \sqrt{3}(e^{-k\tau} - 1) - k\sqrt{3}[(e^{-k\tau} - 1)q(\infty) + q(\tau)] \tag{II.7}$$

and by means of it constructed expansions of the functions F_λ and $\tilde{F}_\lambda$:

$$F_\lambda(\tau, \eta) = F_1(\tau, \eta)(1 + k\eta) - \eta\sqrt{3}\left(1 - \frac{e^{-k\tau}}{1 - k\eta} + e^{-\tau/\eta}\frac{k\eta}{1 - k\eta}\right)(1 - kq(\infty)) -$$
$$k\eta\sqrt{3}[q(\tau) + \eta(1 - e^{-\tau/\eta})], \tag{II.8}$$

$$\tilde{F}_\lambda(\tau, \eta) = \tilde{F}_1(\tau, \eta)(1 - k\eta) + \sqrt{3}\eta(1 - kq(\infty))\left(\frac{e^{-k\tau}}{1 + k\eta} - 1\right) - \sqrt{3}k\eta[q(\tau) - \eta]. \tag{II.9}$$

The expansion of the function $C_\lambda(\tau)$ follows by substitution of the expression (II.9) in (24).

Note that the last relations for s_{-k} and h_{-k} in (II.5) and (II.6) are exact. In the conservative case, they have the form

$$\tilde{s}_0 = \sqrt{3}[S_1 - q(\tau_0)], \quad h_1 = \frac{1 - h_0}{2}\tau_0 - \tau + \zeta, \tag{II.10}$$

where $f_1 \equiv \int_0^1 \mu f(\mu)\,d\mu$.

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