Nonlinear transfer problems and the renormalization group

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Several developments of the ideas of Ambartsumyan on invariance in the theory of radiative transfer[1] have led to the construction of a rigorous, and at the same time physically clear mathematical formalism[2] for the solution of transfer problems in plane-parallel media. The formalism can also be applied to the solution of nonlinear problems; such a possibility was first pointed out by V. A. Ambartsumyan[3] for the case of reflection from a one-dimensional medium. As an illustration of this formalism, we consider a more general nonlinear problem of the internal light field in a one-dimensional medium.[2]

1. Let a stationary monochromatic radiation field of intensity u be incident from outside the boundary of a semiinfinite one-dimensional medium. We let $Y(\tau, \mu)$ be the intensity of the ingoing radiation (directed into the depth of the medium) as a function of the geometrical depth τ. According to the reversibility principle of optical phenomena, $Y(\tau, \mu)$ is also the intensity of the outgoing radiation from the semiinfinite medium, under the condition that at depth τ there be a primary stationary current u directed outward from the medium.

We split a layer of thickness τ into two layers with thicknesses t and $\tau - t$. Then simple physical considerations allow the derivation of the following semigroup property for the function Y:

$$Y(\tau, u) = Y(\tau - t, Y(t, u)). \tag{1}$$

We take the limits as t and $\tau - t$ go to zero, successively, thereby obtaining the pair of differential equations

$$\frac{\partial Y(\tau, u)}{\partial \tau} = -G\,(Y(\tau, u)), \tag{2}$$

$$\frac{\partial Y(\tau, u)}{\partial \tau} = -\frac{\partial Y(\tau, u)}{\partial u}\,G(u) \tag{3}$$

with the initial condition

$$Y(0, u) = u. \tag{4}$$

We denote by G the generating operator of the semigroup

$$G(u) = -Y'_\tau\,(0, u). \tag{5}$$

Equation (2) is the nonlinear analog of the basic equation of transfer theory, while (3) is the same equation in a different notation. By equating the right-hand sides we get the following differential equation

$$\frac{\partial Y(\tau, u)}{\partial u} = \frac{G(Y(\tau, u))}{G(u)},$$

which is the so-called commutation relation.[2]

2. If we introduce the transformation

$$V(u) = \int \frac{du}{G(u)}, \tag{6}$$

then the solution of (2) can be written in the form

$$Y(\tau, u) = V^{-1}(V(u) - \tau), \tag{7}$$

where V^{-1} is the inverse function to V.

Consider the example $G(u) = ku/1 + \beta u$ which corresponds to the case when the intense radiation induces a "translucence" in the medium. The solution (7) reduces to a transcendental equation

$$Ye^{\beta Y} = ue^{\beta u - k\tau}. \tag{8}$$

In the linear case of small intensities $u \ll 1/\beta$ we have $Y(\tau, u) = ue^{-k\tau}$.

Thus the general solution (7) to the problem of finding the intensity Y, which depends on two variables (τ, u), completely reduces to knowledge of a function of one variable G(u). The solution $Y(\tau, u)$ itself only depends on the combination of variables $V(u) - \tau$. In other words, we have succeeded in lowering the number of arguments in the function Y.

When $\tau \to \infty$, expression (7) reduces to the solution of the Milne problem: $Y(\infty, u) = V^{-1}(-\infty)$.

The function G(u) determines to what extent the intensity of the radiation outgoing from the semiinfinite medium is decreased upon the introduction of an infinitesimally thin layer placed in front of the medium, G(u) is easily constructed physically from the local optical properties of the medium for a given model, and is expressed in terms of the analog of the Ambartsumyan φ-function for the nonlinear case. The function G(u) is also the simplest to determine experimentally for the problem of an internal light field.

3. Muradyan has pointed out to us the superficial similarity of the relations given above (and in Ref. 2) and the known equations of the renormalization group method (RG) in quantum field theory. The mathematical analog can be established completely with the help of the transformation

$$\tau \to \ln x,\quad t \to \ln t,\quad u \to h,\quad Y(\ln x, h) \to \bar{h}(x, h) \tag{9}$$

which corresponds to the notation of Ref. 4 (p. 388). The semigroup property (1) takes the form

$$\bar{h}(x, h) = \bar{h}\left(\frac{x}{t},\ \bar{h}(t, h)\right), \tag{10}$$

which is the RG equation for the invariant charge $\bar{h}$ in a single-charge massless quantum field theory (formula

49.4 of Ref. 4). Here $x = k^2/\lambda$, where k is the momentum, λ is the square of the normalization momentum. Using the notation (9) we find that (2) transforms to the Lee differential equation, while (3) transforms to the Ovsyannikov–Kallan–Simanchik equation. The initial condition (4) becomes the normalization relation $\bar{h}(1, h) = h$. Expression (6) is the analog of the Gell-Mann–Low equation (Eq. 49.14 of Ref. 4). The normalization momentum corresponds to the position of the boundary of the semiinfinite medium; the choice of both is arbitrary. The solution of the Milne problem determines the so-called "bare" charge. Below we shall discuss the analog between the critical thickness of the layer and the "virtual pole" of the invariant charge.

In the language of light scattering theory, the RG equation corresponds to a "random walk" of the invariant charge in momentum space (more precisely, in the space of the logarithm of the momentum). The solution of this equation determines the value of the invariant charge $\bar{h}$ for a given value of the momentum.

4. In the theory of radiative transfer, an important problem is the determination of the number of scattered quanta in the medium. For the model considered above, we let $N(\tau, u)$ be the total number of scatterings experienced by all quanta outgoing from the semiinfinite medium, per unit time.

Again we divide the layer τ into two layers $\tau - t$ and t. Then it is trivial to obtain the relation

$$N(\tau, u) = N(t, u) + N(\tau - t, Y(t, u)). \tag{11}$$

From this follows the pair of differential equations

$$\frac{\partial N(\tau, u)}{\partial \tau} = \sigma(Y(\tau, u)), \quad \frac{\partial N(\tau, u)}{\partial \tau} = \sigma(u) - \frac{\partial N(\tau, u)}{\partial u} G(u) \tag{12}$$

with the initial condition $N(0, u) = 0$. Here we put

$$\sigma(u) = N'_\tau(0, u). \tag{13}$$

From (12) we find

$$N(\tau, u) = \int_0^\tau \sigma(Y(t, u))\, dt. \tag{14}$$

If we again use the notation (9) and put

$$N \to \ln s, \tag{15}$$

then (11) transforms to the RG equation

$$s(x, h) = s(t, h)\, s\left(\frac{x}{t}, \bar{h}(t, h)\right) \tag{16}$$

for the single-argument Green's function. The solution (14) of this equation is also known in field theory.[4] In the language of transfer theory ln s is the number of interaction events between an electron and virtual photons for a given value of the momentum transfer.

5. The analog between the equations of the transfer problem and the RG is actually much more general. Instead of monochromatic quanta, we consider the case of two frequencies, when in each scattering event, a transformation of quanta from one frequency to those of the other is possible. Let the primary currents for the two frequencies be u and v. Then the outgoing currents at the two frequencies $Y_u(\tau, u, v)$ and $Y_v(\tau, u, v)$ obviously satisfy

$$\begin{aligned} Y_u(\tau, u, v) &= Y_u(\tau - t, Y_u(t, u, v), Y_v(t, u, v)), \\ Y_v(\tau, u, v) &= Y_v(\tau - t, Y_u(t, u, v), Y_v(t, u, v)) \end{aligned} \tag{17}$$

with the initial conditions $Y_u(0, u, v) = u$, $Y_v(0, u, v) = v$.

In quantum field theory, the analog of these equations is the RG equation for the invariant charges in a two-charge model.

In the general case of polychromatic scattering, the equations for the nonlinear problem in a semiinfinite one-dimensional medium are given by the system

$$Y_k(\tau, \{u\}) = Y_k(\tau - t, \{Y(t, \{u\})\}), \quad k = 1, 2, \ldots, n. \tag{18}$$

They are analogous to the RG equations of a multicharge model in the theory of interacting fields with n coupling constants (invariant charges).

Recall that the nonlinearity of the problem considered here is expressed by the fact that the optical properties of the medium depend on the intensity at different frequencies at a given point, and in particular, the dependence changes with depth. In the stationary problem the optical properties are ultimately determined by the primary currents $\{u\}$. The case of two frequencies (17) is treated here only formally; physically, quanta of a third frequency are also required, so that the rigorous description of this case must be based on Eq. (18) with n = 3.

6. The analog between the solutions of the transfer equation and the RG is not limited to only the examples treated above. These describe the asymptotic limit of semiinfinite media, and this corresponds to the ultraviolet asymptotic form of the Green's function, i.e., to the limiting case of large momenta (or zero mass).

We now consider the problem of the diffusion of light in a layer of finite thickness $|\tau_0| + \tau$. In the monochromatic case the function $y(\tau, \tau_0, u)$, which is analogous to $Y(\tau, u)$ above, is found to satisfy the relation

$$y(\tau, \tau_0, u) = y(\tau - t, \tau_0 - t, y(t, \tau_0, u)). \tag{19}$$

Equation (19) is the analog of the RG equation for the invariant charge in the general case of nonzero mass m [in fact we have $\tau_0 \to \ln(m^2/\lambda)$].

For polychromatic scattering

$$y_k(\tau, \tau_0, \{u\}) = y_k(\tau - t, \tau_0 - t, \{y(t, \tau_0, \{u\}), \quad k = 1, 2, \ldots, \tag{20}$$

which is the analog of a multicharge model and nonzero mass. For example, in the two-charge model, the RG equation for the invariant charges are given by (48.12) and (48.13) of Ref. 4.

7. In practice, it is not always possible to work with analytic expressions; in cases where numerical solution is necessary, it is convenient to use the semigroup relation (1), and determine the solution recursively in the depth τ. This is done in a particularly simple manner graphically, with $Y(u)$ diagrams for fixed discrete values of τ. In some cases, when the currents are not very large, one can effectively combine perturbation theory

with the renormalization equations, as is commonly used in quantum theory.

Following this method, we expand $Y(\tau, u)$ in the form

$$Y(\tau, u) = u + f(\tau)u^2. \tag{21}$$

Calculating G(u) from (5), V(u) from (6) and substituting into (7) we obtain

$$Y(\tau, u) = \frac{u}{1 - \alpha u \tau}, \quad \alpha = f'(0). \tag{22}$$

In transfer theory, choice of the coefficients in the linear term in (21) to be unity, corresponds to pure or conservative scattering. In the case of light quanta, the exact solution of the problem in this case must be $Y(\tau, u) = u$ for all τ and u. However under the condition that $f(\tau)$ be positive (spinor electrodynamics) we have $Y(\tau, u) > u$, i.e., the medium generates particles (neutrons), and the positive value of α means that the cross-section for splitting the nucleus increases with increasing u. An example is a plutonium reactor with fast neutrons.

Expression 22 diverges for $\tau = \tau_{cr} = 1/\alpha u$. This corresponds to a critical size of the reactor (in the nonlinear case it is dependent on the intensity). It is true that the value τ_{cr} is rather large, but here the important point is that there exists a finite value for the critical thickness of the medium. We point out in passing that this implies that the problem cannot be stationary.

In quantum field theory, a singularity in the invariant charge h follows from the analog of the Landau–Abrikosov–Khalatnikov[4] formula. This is the "virtual" pole described by Landau–Pomeranchuk–Fradkin (see Sec. 50.2 of Ref. 4) and leads, as is well-known, to a series internal contradiction of the theory. From the treatment of criticality given above, the appearance of a "virtual pole" is unlikely to be due to errors in the perturbation theory, as is usually claimed. The existence of a critical thickness is the result of the relation $Y(\tau, u) > u$ over sufficiently large intervals of the depth τ, as in the initial representation (21). In this case, we find that the number of scatterings in a layer of critical thickness also diverges (the "infrared catastrophe"). For a semiinfinite medium which can generate particles, the functions Y and G must in general be taken as complex.[2] If $f(\tau)$ is negative (quantum chromodynamics) then criticality does not occur (asymptotic freedom).

8. The above mathematical analog between the solutions of nonlinear problems in transfer theory and the RG equations of quantum field theory allow one to carry over the methods and results from one theory to the other. For example, the general Ovsyannikov solution[4] of the RG equation is also valid in nonlinear transfer theory. On the other hand, in transfer theory, explicit solutions of certain nonlinear problems can be found using other methods, whereas in quantum electrodynamics the practical solution of the corresponding problems is based on perturbation theory. We also do not rule out the possibility that this analog has a deeper physical foundation, whose recognition would shed light on questions concerning these physical processes.

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