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Editors
D.V. Shirkov
D.I. Kazakov
A.A. Vladimirov

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PREFACE

International conference "Renormalization Group-86" was held at Dubna, USSR in August 1986. The subject of the conference was numerous applications of the ideology and technique of the renormalization group in various fields of theoretical physics.

Renormalization group was discovered in 1953 while studying the structure of renormalizations in quantum field theory. At first, it was used to analyse ultraviolet behaviour in different QFT models. In particular, with the help of RG the phenomenon of asymptotical freedom in QCD was found. At the beginning of the seventies the RG method was successfully applied in the theory of phase transitions to the calculation of critical exponents. During the next decade, the RG method found application in various far separated fields of physics such as the theory of turbulence, polymer physics, transfer theory, percolation and others. In recent years, the renormalization group ideology has been used in the theory of dynamical chaos. So, the RG method appears to be a universal technique used to handle the singularities in a wide class of complicated physical problems.

The aim of the conference was to gather the specialists from different fields of theoretical physics who use the renormalization group ideology in their research.

At the conference 10 invited review talks and more than 30 original reports were given. The present volume contains all the review talks and half of the original reports. The editors' aim was to choose those original papers which are interesting from the point of view of renormalization group ideology.

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Editorial board

RENORM-GROUP ANALOGIES IN ASTROPHYSICS

M. A. Mnatsakanian

Byurakan Astrophysical Observatory, Ac. Sci. Armenian SSR, Yerevan State University, USSR

C o n t e n t s

Analogies by themselves, of
course, do not solve anything, however
they make oneself feel at home.
Sigmund Freud

Application of renorm-group concepts and methods to various fields of modern physics has also affected to some extent the astrophysics. It is true, however, that origination of renorm-group analogies in the main body of theoretical astrophysics, the radiation transfer theory, had more specific and independent character and was due to Ambartsumian's invariance concepts. These analogies, developed mainly in the authors researches, are discussed in the present paper.

New semi-group relations describing the solutions of non-stationary and non-linear radiation transfer problems are obtained in the paper. They result in efficient research tool which is different from classical methods even in stationary and linear cases. A number of particular examples and their solutions are also considered.

Advantages of semi-group theory are discussed. The role of Ambartsumian's principle of invariance is shown, and relation with Shirkov's functional self-similarity is revealed. Multiple mathematical analogies are established with renormalization group and problems of quantum field theory. On basis of these analogies, an attempt is made to extend some physical results from one theory to another.

In some sense our semi-groups generalize the semi-group method of Kolmogorov, Feller and Chapman used in the Markov processes theory. They are extended to non-linear process, while in the linear problems our semi-groups are applied in the case when non-monotonic random parameter of depth is considered instead of monotonic

time parameters.

Therefore we discuss the application of semi-groups in a number of problems originating in various fields of science(synergetics) and show their efficiency in mathematical modelling problems.

Finally, an astrophysical example is considered. We solve the inverse problem of finding the distribution function for stellar star flares frequency in aggregates. Possibility of applying the semi-group method to problems of stellar star systems dynamics is discussed. Papers of other authors considering the application of renorm-group method in astrophysical problems are mentioned.

In all these problems, and especially in mathematical modelling, the semi-group approach suggests new possibilities of their efficient analysis, particularly by means of analog modelling and adoption of methods and results.

1. SEMI-GROUPS IN RADIATION TRANSFER THEORY

Radiation transfer theory forms a foundation of theoretical astrophysics. It studies radiant energy transfer processes in media of various structure and form. Close to that theory are the problems of neutron transfer and many other problems of nature.

Mathematically the radiation transfer problem represents a boundary problem. Its solution by classical methods is the analysis of differential equations with initial and boundary conditions. Introduction of semi-group description in the radiation transfer problems changes the approach to such problems and makes possible their easier analysis.

1.1 Non-Linear and Non-Stationary Problem of Polychromatic Light Scattering

1.1.1 Plane homogeneous layer. Let the upper boundary of a plane medium, having geometric thickness τ_0, be illuminated from time $t=0$ by polychromatic light fluxes $u_k(t)$ at various frequencies ω_k. In the general case, the non-linearity of problem to be considered is due not only to polychromatic character of radiation (i.e. "interaction" of quanta having different frequencies), but also to radiation power at specific frequencies, which may cause changes in local optical properties of the medium (including the dependence on direction of induced radiation).

We are interested in radiation flux y_k at a time $t>0$ inside the medium at geometric depth τ in the direction of increasing τ. This quantity represents a unique functional of boundary fluxes vector $\{u_m(t)\}$ over the time interval $(0,t)$. We shall denote this fact by a tilde sign $\sim$: $\{y_k\}=\{y_k(\tau,\tau_0,t,\{\tilde{u}_m(t)\})\}$, or in a more simple form where indices are omitted: $y=y(\tau,\tau_0,t,\tilde{u}(t))$.

Under the condition of existence and uniqueness of problem solution the radiation flux $y(\tau_1,\tau_0,t,\tilde{u}(t))$ at a depth τ_1 (which represents the result of multiple quantum scattering in the whole medium and re-distribution in quantum characteristics: frequencies, directions, etc) at the same time serves a boundary flux to the remainder part of the medium with thickness $(\tau_0-\tau_1)$. Therefore, the fluxes at a depth $\tau_1+\tau_2\leq\tau_0$ are unambiguously determined by their values at depths τ_1 and τ_2 by means of recurrent relations:

$$y(\tau_1+\tau_2,\tau_0,t,\tilde{u}(t))=y(\tau_2,\tau_0-\tau_1,t,\tilde{y}(\tau_1,\tau_0,t,\tilde{u}(t))). \qquad (1)$$

According to the expression (1), and from physical considerations we have

$$y(0,\tau_0,t,\tilde{u}(t))=u(t). \qquad (2)$$

Semi-group relations (1) are true for any quantum re-emission and free transition times, as well as for any law of elementary scattering act. Corresponding characteristics are implicitly taken into account by means of infinitesimal generating operator (or generator) of the semi-group:

$$g(\tau_0,t,\tilde{u}(t))\equiv y'_\tau(0,\tau_0,t,\tilde{u}(t)). \qquad (3)$$

The third argument t in (1) enters as a parameter.

For a semi-infinite medium the semi-group relation is simplified, and for the similar quantity $Y(\tau,t,\tilde{u})$ becomes:

$$Y(\tau_1+\tau_2,t,\tilde{u}(t))=Y(\tau_2,t,\tilde{Y}(\tau_1,t,\tilde{u}(t))), \qquad (4)$$

$$Y(0,t,\tilde{u}(t))=u(t). \qquad (5)$$

1.1.2 Spherical layer. The relations given above are true for a more complex medium geometry, in form of homogeneous spherical layer of thickness τ_0, when fluxes $y(\tau,\tau_0,t,\tilde{u}(t))$ are searched (in solid angle subspace corresponding to increasing τ) at a depth τ measured from the illuminated boundary. Curvature radius of the second surface serves as a parameter in these relations. Therefore, it is omitted. In the case of semi-infinite spherical layer, either the filled sphere or infinite space with cavity this radius is absent.

Due to the formal similarity of semi-group relations for plane layer of thickness τ_0 and sphere of radius τ_0 some of their characteristics should be identical (e.g. the critical size).

1.1.3 Presence of reflecting interface. The second boundary of a plane or spherical layer may be adjacent with a medium having arbitrary given properties or be illuminated by a flux ϑ. In this case all relations given above will be valid, while the quantity ϑ enters as a parameter.

1.1.4 Discrete random walk. In essentially the same manner one may analyse the problems of random walk over discrete lattices. Here non-linearity may express, for example, the same psychology in the ruin problems, where the player strategy depends on the stake value. Note the form of solution for the semi-infinite lattice in the stationary case:

$$Y(m,u) = M(M\ldots M(u)\ldots) = M^m(u) \qquad (6)$$

where $M(u) \equiv Y(1,u)$, $m = 1,2,3,\ldots$ are lattice codes.

1.1.5 Optical reversibility. If one properly formulates the generalization of reversibility principle for optical phenomena, including the non-linear case (where dependence of absorption factor on radiation intensity should be taken into account), then the quantity y may also describe the emergent radiation in presence of internal primary fluxes.

The amount of radiation itself, emerging from the medium will satisfy the similar semi-group relations, under the condition that the "initial" internal fluxes standing for $u(t)$ are in their turn caused by scattering processes of radiation coming from deeper layers, i.e. from infinity (Miln problem) in the case of semi-infinite medium, or through the other surface of layer τ_0 (transmission problem) in the case of medium with finite optical thickness.

1.2 Linear Problem of Incoherent Scattering.

1.2.1 Semi-infinite medium. In the linear problem with re-distribution over frequencies within the spectral line when the principle of superposition for fields is true, one may consider separately the case when semi-infinite medium boundary is illuminated by a pulse-like flux $u(t) = u_0 \cdot \delta(t)$ and construct operatoral relations

$$Y(\tau_1+\tau_2,t) = \int_0^t Y(\tau_2,t-t')\,Y(\tau_1,t')\,dt', \qquad (7)$$

$$Y(0,t) = \delta(t)\cdot I \qquad (8)$$

(where I is identity operator) for the integral operator Y /2/ acting the frequency space, etc. Hence convolution with $u(t)$ distribution in time gives the relation (4). Earlier we have denoted $Y(\tau,t,u) = Y(\tau,t)\cdot u$.

1.2.2 Finite thickness layer. One may write down the relation similar to (7) for the finite thickness layer as well. However in linear problems we prefer a more efficient method of reducing the problem to a more particular and easier problem on semi-infinite medium /1/. For example, we may write for quantities y and z (where z describes descending flux at a depth τ in the direction opposite to the one described by a quantity y) the following relations

$$\begin{aligned} Y(\tau,t) &= y(\tau,\tau_0,t) + \int_0^t Z(\tau_0,t-t')\,z(\tau,\tau_0,t')\,dt', \\ Z(\tau,t) &= z(\tau,\tau_0,t) + \int_0^t Z(\tau_0,t-t')\,y(\tau,\tau_0,t')\,dt'. \end{aligned} \qquad (9)$$

Here Y and Z are semi-infinite medium characteristics. Similar relations are found for solution of any problem considering the finite thickness layer.

Using the Laplace transformation "non-stationary" relations (7)-(9) may be replaced by corresponding "stationary" relations for Laplace transforms. Thus, for example, from relation (7) we obtain

$$\overline{Y}(\tau_1+\tau_2,s) = \overline{Y}(\tau_2,s)\,\overline{Y}(\tau_1,s), \quad \overline{Y}(0,s) = I. \qquad (10)$$

Here transformation parameter s enters by means of infinitesimal generating operator $\mathcal{G}(s) \equiv \overline{Y}'_{\tau}(0,s)$ of semi-group (10).

1.3 Non-Linear Stationary Problem

In a stationary problem, $u(t) = u = Const,\ t \to \infty$ the semi-group relations (1) and (4) are transformed into relations obtained earlier by the author /2/:

$$y(\tau_1+\tau_2,\tau_0,u) = y(\tau_2,\tau_0-\tau_1,y(\tau_1,\tau_0,u)), \tag{11}$$

$$Y(\tau_1+\tau_2,u) = Y(\tau_2,Y(\tau_1,u)) \tag{12}$$

with boundary conditions

$$y(0,\tau_0,u) = u, \quad Y(0,u) = u. \tag{13}$$

Remind that indices are omitted in these expressions so that they actually describe a polychromatic scattering

Setting τ_1 or τ_2 in (12) infinitely small we obtain two generally independent differential equations

$$\frac{dY(\tau,u)}{d\tau} = \mathcal{G}(Y), \quad \frac{dY(\tau,u)}{d\tau} = \frac{dY(\tau,u)}{du}\mathcal{G}(u). \tag{14}$$

In the case of monochromatic scattering in one-dimensional medium their solution is given by the expression

$$\tau = \int_u^{Y(\tau,u)} du/\mathcal{G}(u). \tag{15}$$

For the real physical problem the function $\mathcal{G}$ has the following form /1/: $\mathcal{G}(u) = ku/(1+\beta u)$ and we have from (15) the transcendental equation

$$Ye^{\beta Y} = ue^{\beta u - k\tau} \tag{16}$$

where $1/\beta$ is characteristic value of the flux, from which the non-linear effects are revealed.

We also observe that the total number of scatterings experienced by quanta, moving at a depth τ into the bulk of semi-infinite medium per unit time, satisfy the following equation

$$N(\tau_1+\tau_2,u) = N(\tau_1,u) + N(\tau_2,N(\tau_1,u)), \quad N(0,u) = 0 \tag{17}$$

Its solution is given by the expression

$$N(\tau,u) = \int_0^{\tau} \sigma(Y(t,u))dt, \tag{18}$$

where $\sigma(u) \equiv N'_{\tau}(0,u)$.

In the case of two-frequency quanta that cannot be transformed from one into another (a three-level atom with forbidden transitions between the upper layers) similar solution (obtained by D.V.Shirkov and the author) is given by the following expression

$$Y_1 e^{\beta_1 Y_1 + \beta_2 Y_2} = u_1 e^{\beta_1 u_1 + \beta_2 u_2 - k_1\tau}$$
$$Y_2 = u_2 (Y_1/u_1)^{\nu} \tag{19}$$

where u_1 and u_2 are boundary fluxes at individual frequencies, k and ν are constant characteristics for the linear case.

1.4 Advantages of Semi-Group Approach.

The semi-group approach has significant advantages as compared with the classical methods of transfer theory.

The first and the most important point is that the boundary problem, when semi-group description is used, is reduced to one or series of Cauchi problems. Two differential equations following from relations (11) and (12), such as (14) are essentially different (even in the linear problem) from the classical transfer equation. The latter has in its right hand part a source function involving both ascending and descending radiation fluxes. Meanwhile in our case we separately determine only the ascending radiation y. Descending radiation z is completely determined by y.

The second point is that we are able to solve the relations (11) numerically, by recurrent procedure with finite steps in τ ,e.g. doubled steps. Meanwhile, the solution of differential equation requires infinitely small steps $d\tau$. The latter procedure also results in significant accumulation of computing errors.

The third point is that experimental determination of generating operator $G(u)$ is difficult, since one has to find the derivative $dY(\tau,u)/d\tau\,|_{\tau\to 0}$ as accurately as possible. Alternatively one may measure, say, the quantity $Y(\Delta,u)$ for any finite Δ , then the semi-group relation makes possible to find the exact value of $Y(\tau,u)$ for any τ multiple of Δ .

Other advantages of semi-group approach are: possibility of carrying itterations and analytical aspects of researches. For the linear problems these advantages were shown in detail in the author's papers [2,3]. Analytical abilities provided by reduction method in linear problems were demonstrated in papers [1,3]. We observe here that knowledge of stationary solutions is even unnecessary for solution of non-stationary problems. It is sufficient to know their Laplacian transforms, found recurrently by means of relation (10).

Our derivation of semi-group relations is based on the following considerations: solution to the transfer boundary problem is unique; if a part of (homogeneous) layer is removed, then the integral optical characteristics of the remainder layer will have the same functional dependence on the layer thickness as for the initial layer (this is the formulation of Ambartsumian's principle of invariance [4]); the quantity y for the cut-off and complete layer depends on the same number of arguments; one should consider the quantity (y is unique) which is transformed into flux u on the layer boundary,

the latter being given as a boundary condition.

When these conditions are violated, it is either impossible to construct the semi-group relation (say, for the quantity z), or such relation will be inefficient, since the quantities present in that relation will depend on additional arguments. The latter fact means that the chosen quantity is not invariant in the true sense of that concept.

These circumstanses make possible finally to reduce the number of arguments in the problem solution by means of obtaining the semi-group relations. Thus, for example, the general solution (15) actually means that the function $Y(\tau,u)$ depends on a single combination of its two arguments:

$$Y(\tau,u)=V^{-1}\left(V(u)-\tau\right) \qquad (20)$$

where $V(u)$ is arbitrary function (determined by the properties of G).

By its physical meaning, the infinitesimal operator G (or g) determines those changes in quantity Y (or y) which are due to addition of infinitely thin layer to the medium. In this sense the operator G expresses the invariance ideas of V.A.Ambartsumian, therefore we call it invariance operator. In each particular problem this operator is constructed on basis of additional physical consideration about the local optical properties of the medium. Without specification of that quantity, the general solution determined by the semi-group relation does not involve any information on elementary scattering act characteristics.

2. ANALOGIES IN QUANTUM FIELD THEORY

Results presented in the previous part of our report are of special importance because of the following interesting observations. It turned out/5/ that many of mathematical semi-group relations, and their consequences published in our paper/2/ were known long ago in the entirely different research field - theory of interacting quantized fields QFT /6/. Below we shall discuss these analogies in detail, and try to draw them at a level of corresponding physical processes.

Mathematical sense of these analogies, which exist also in the other fields of science, has been rigirously formulated by D.V.Shirkov /7/, in the functional self-similarity concept introduced by him.

2.1 Analogies with Renormalization Group.

Our semi-group relations tirn out to be completely similar to the renormalization group equation in QFT, except that they are presented in the additive form. In order to transform them into multiplicative form one has to make a change of variables:

$$\tau \to \ln x\,,\quad u \to e^2 = h\,,\quad Y(\tau, u) \to \bar{h}(x, h). \tag{21}$$

Then the equation (12) will be transformed into renorm-group (RG) equation

$$\bar{h}(x,h) = \bar{h}\left(\frac{x}{t}, \bar{h}(t,h)\right)\,,\quad \bar{h}(1, e^2) = e^2 \tag{22}$$

for invariant charge in the single-charge massless theory. The function $G(u)$ represents the analog of so-called β-function, while equations (14) are analogs of Lea equation and Ovsyannikov-Callan-Symanchik equation. Meanwhile the solution (15) represents the analog of Gell-Mann and Low equation /8/. In that analogy normalization point or substraction pulse $x^2 = p^2/\lambda^2$ corresponds to the boundary of medium. Milne problem solution corresponds to the light scattering on a "stripped" charge.

Redenoting $N \to \ln s$, from relation (17) we obtain the RG equation

$$s(x,h) = s(t,h)\, s\left(\frac{x}{t}, \bar{h}(t,h)\right) \tag{23}$$

for the single-argument Green function (see, /6/, §47).

We have considered above the scalar equation, i.e. monochromatic scattering in the transfer theory. In the case of polychromatic scattering equations (12) represent exact analogs of equations for invariant charges (bond constants) in the multi-charge model of QFT. The double-charge model has been first considered in paper/9/. If one considers the transfer problems for a layer of finite thickness τ_0, and introduces $\tau_0 \to \ln(m^2/\lambda^2)$ together with expressions (21), then the semi-group relations(11) will be transformed into similar RG equations in a general model with non-zero masses. Meanwhile the asymptotic case of non-zero masses, or extremely large momenta, corresponds to semi-infinite medium.

It would be interesting to find the particular analogs in GFT of semi-group relations/1/, describing non-stationary transfer problems, as well as analogs of reduction method relations (9).

2.2 On Physical Analogs.

Mathematical analogies, existing between the non-linear problem solutions in transfer theory and RG equations in QFT, make possible the transfer of solution methods and results from one theory into another. Thus, for example, the general solutions of RG equations obtained by Ovsyannikov /10/ are applicable to the non-linear transfer theory. Now we would like to consider the physical aspects of these analogies using the example of electron potential scattering.

Charge interaction effect in the QFT is due to transfer (or exchange) of momentum by virtual photon. This phenomenon is accompanied by multiple (same as in the transfer theory) creation and annihilation processes.

Instead of light quantum scattering on atom of matter, i.e. instead of absorption and emission of quantum by atom, in GFT takes place, speaking figuratively, the absorption and emission of electron by virtual photon (or more correctly, of electron-positron pair). Virtual photon will play the role of excited atom in QFT. Meanwhile, the medium atom. on which the light is scattered, may be considered as a virtual electron. Thus, when we pass from QFT to the light scttaring theory, electrons and quanta exchange their roles.

We turn now to the corresponding Feynman diagrams. In Fig.1 one may see the corresponding diagrams for diffusive reflection of quantum from the medium (Fig.2). In the latter case the multiple scattering effect is determined by a reflection factor R , being the analog of form-factor f .

We may see that electron-positron pair corresponds to every act of scattering from the layer, while atomic excitation and emission of quantum correspond to the virtual photon in QFT.

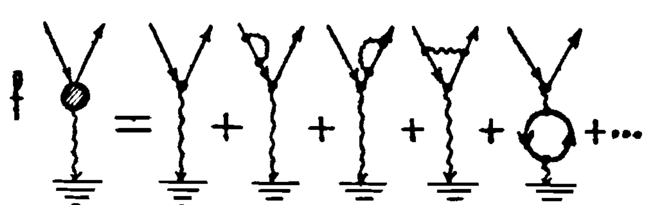

Fig.1 Feynman diagrams for electron scattering by a potential

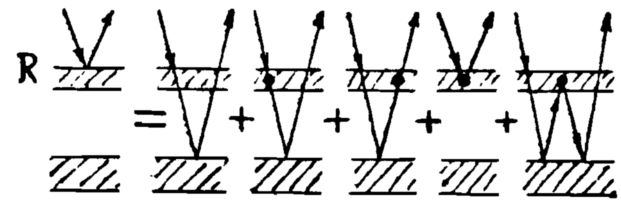

Fig.2 Diagrams corresponding to application of Ambartsumian's principle of invariance in reflection problem

2.3 Some Inferences by Analogy

Due to analogies, it becomes possible to adopt certain qualitative inferences. Some of them are considered in a popular paper/11/ by N.N.Bogolyubov and D.V.Shirkov. These inferences are based on renorm-group method/12/.

Following that method, we present $Y(\tau,u)$ in form of decomposition:

$$Y(\tau,u) = u + f(\tau)u^2. \tag{24}$$

After calculating $G(u)=Y'_\tau(0,u)$ and inserting into expression (15) we obtain

$$Y(\tau,u)=\frac{u}{1-\alpha u\tau}, \quad \alpha \equiv f'(0). \tag{25}$$

When $f(\tau)>0$ (spinor electrodynamics) we have $Y(\tau,u)>u$ i.e. the medium generates particles (neutrons). Therefore the existence of "ghost pole" in expression(25) may be compared with a critical thickness concept existing in transfer theory. If $\alpha<0$ (quantum chromodynamics), then the divergencies of $Y(\tau,u)$ when $\tau\to\infty$ no more exist, and the case $Y(\tau,u)\to 0$ corresponds to asymptotic freedom in QCD.

Consider now the charge renormalization problem in order to specify the behaviour of electron efficient charge $Q(r)$ at small distances r (or large momenta), which enters the Coulomb law: $Q(r)/r^2$. It is usually assumed that this charge is equal to the observed value e at a distance R_o equal to the Compton wavelength of electron. Inside the cloud of radius R_o (we accept it as a unit) there take place the processes of vacuum polarization and virtual photon decay into electron-positron pairs. In logarithmic variables $\ln r=-\tau$ the value R_o corresponds to the medium "boundary", while the cloud is transformed into a half-space $\tau>0$. Hence, by analogy with scattering problem, one may conclude that electron-positron pair "scattering" process by virtual photons has Poisson Law character (no matter what power has r in denominator of the Coulomb law). Only in that case we shall obtain the semi-group relation (12) with corresponding renorm-group equation

$$Q\left(\frac{r}{R_o},e\right)=Q\left(\frac{r}{R},Q\left(\frac{R}{R_o},e\right)\right),\quad Q(1,e)=e.$$

We conclude that the distance between the sequential processes in QFT is uniformly distributed. Therefore, the logarithm of that distance will have a distribution which is universe to logarithmic, i.e. an exponental distribution. This law corresponds exactly to the Poisson distribution of optical depth before the first quantum scattering by the atoms of medium. Thus the most significant condition which determines the semi-group relation is satisfied in our analogy.

2.4 D.V.Shirkov's Functional Self-Similarity.

Semi-group relations do not involve the details of light scattering process, since they enter in the G function. In the same manner, the RG equations (obtained on basis of rather specific concepts and arguments concerning the ultraviolet divergencies renormalization procedures) do not depend on details of a particular quantum field model (the details being involved in the β-function). Similar situation is with equations described below in examples taken from non-linear mechanics and ecology.

Such universal character of semi-group equations reflects, according to D.V.Shirkov/7/, the transitivity of system's physical characteristics with respect of specification method of their initial or boundary conditions. This fact served a reaspn for D.V.Shirkov to introduce the special concept of "functional self-similarity", which agrees with the observation made in section 4 about the statement of the boundary problem. The concept was, particularly, an argument in favour of application of a general semi-group relation given below (see 3.1) to Cauchi problems as well.

Self-similarity concept is a generalization of similarity transformation (or scaling) concept, which is used in many fields of mathematical physics, and expresses the invariance property with respect to the power transformation of parameters/13/.

D.V.Shirkov/7/ makes an essential distinction between dynamic equations describing the system variation in time, and evolutional equations over some other "evolutional" parameter. Examples of evolutional parameters are the depth τ in transfer problems, normalization momentum or distance x in QFT. The dynamic parameter in the first case is, for example, the path length passed by the quantum. Existence of both types of equations in each of these research fields significantly increases its efficiency.

3. PROBLEMS FROM OTHER RESEARCH FIELDS

Now we shall discuss the application potential of semi-group relations to problems of other fields of science. Among them are purely mathematical problems concerning generalization of Markov processes theory to include non-linear and other cases, problems of synergetics and mathematical modelling, as well as some astrophysical problems.

Our main purpose is to prove the advantage of applying the group relations instead of differential equations

3.1 Mathematical Analogies and Generalizations

The Cauchi problem

$$\frac{dY(t)}{dt} = G(Y) \quad , \quad Y(0) = Y_0 \tag{26}$$

according to Bellman and Cook[14] is a source of inexhaustibly rich mathematical thought. Indeed, it describes an incredible number of problems in various fields of science, representing the expression of deterministic principle used for description of physical system whose initial state is given at a time $t=0$.

Clearly, the solution $Y(t,Y_0)$ in its existence and uniqueness domain, with $Y(0,Y_0)=Y_0$ satisfies the following semi-group relation of type (12):

$$Y(t_1+t_2,Y_0) = Y(t_1,Y(t_2,Y_0)). \tag{27}$$

We observe that in difference to the considered case representing originally the Cauchi problem, the semi-group relation (12) has been obtained in a more complex boundary problem. Therefore, the semi-group (27) in that respect is more or less trivial. Nevertheless, it provides the new abilities to analyse the problem (26).

Thus, particularly, instead of analysing the equation (26) we may, according to relation (27), analyse another equation, an analog of the second equation (14)

$$\frac{dY(t,Y_0)}{dt} = \frac{dY(t,Y_0)}{dY_0} G(Y_0) \; , \quad Y(0,Y_0) = Y_0 \tag{28}$$

as well as the generalized commutativity relation (see below)

$$G(Y) = \frac{dY(t,Y_0)}{dY_0} G(Y_0) \tag{29}$$

which follows from the comparison of equations (28) and (26).

New capabilities of problem (26) analysis on basis of relations (27)-(28) are due to knowledge of behaviour of function $G(Y_0)$, determined by the condition $G(Y_0) = dY(t,Y_0)/dt \,|_{t\to 0}$ specified with respect to a set of initial values Y_0 .

Commutativity relation (29) is not actually a trivial consequence of equation (26), since here, generally speaking, we deal with the operator equation of type (12). Even in the simplest linear case

$$Y(t,Y_0) = Y(t)\cdot Y_0 \tag{30}$$

the commutativity relation

$$GY = YG \tag{31}$$

involves a complete information about the desired quantity Y [2] and results in its determination by a much simpler way. To confirm this statement we refer to numerous examples in transfer theory[2] where the exact solution Y is easily found from a single relation (31).

Consider now the stationary Markov processes, whose theory is developed in the linear case. Here the analysis tool is known to be the semi-group Kolmogorov-Chapman equation /15/ which is similar to the equation (27). At the same time the equations (26) and (28) represent analogs of forward and backward equations of Kolmogorov and Kolmogorov and Feller.

Compare now the situation with a stationary (!)radiation transfer theory. In the latter case we have the depth variable τ instead of time variable. Notably, the diffusing quantum undergoes the abrupt walk in time, and it may be present at various depths in the medium. Therefore the depth is not a monotonously increasing parameter in process of quantum walk. If the walk were only one-sided, i.e. with a δ shaped scattering indicatrix, but of course, abrupt and with various probabilities for different free path lengths, then we would obtain a real analogy of Markov process development in time. However, we deal with a more complicated case of "walking" parameter, and thus the semi-group description of transfer process represents a specific generalization of the similar Markov process description to the case when a"walk" parameter of depth in the layer is used instead of monotonous time parameter.

In order that this difference be more distinct we have to pay attention to the form of semi-group relation (7) for linear transfer process in the case of its non-stationarity. The corresponding semi-group description in Markov processes is made in time, while in our case such description is carried in depth, and in time we have a convolutional type integral.

In this generalization it is essential that discontinuous parameter, such as describing in space or quantum frequency, cannot serve as evolutional parameter,

such as τ .

Meanwhile, the relations (4)-(6) and the following differential equations may serve as generalization of Kolmogorov, Feller and Chapman method in Markov processes theory to include the corresponding non-linear processes.

3.2 Mathematical Modelling Aspects

The possibility of semi-group description should significantly simplify the researches in mathematical modelling of various processes and events. From the arguments given in section 3.1 it becomes obvious that establishment of semi-group properties in various problems does not represent a principal difficulty. There exists a class of non-linear problems that are analysed by entirely new research field, the synergetics /16/. It includes the processes going in lasers, problems in mechanics, ecology, sociology and many others. The aim of synergetics is to analyse the self-organization phenomenon, especially the "phase" transition phenomena in non-linear systems

Corresponding processes admit the semi-group description, which, however, is not used, as far as we know. Here, we shall give two such examples refering to non-linear mechanics and ecology.

3.2.1 <u>Example from non-linear mechanics</u>. The problem concerning the static equilibrium of a homogeneous strongly deformed elastic line loaded by distributed force P and torque H is solved on basis of their balance at very infinitely small portion of the line, which results in the following equation /17/

$$\frac{d^2\zeta}{ds^2} = -\frac{P(s)}{H(s)}\sin\zeta \qquad (32)$$

where s is the length measured along the line, ζ is an angle tangent to the line at a given point.

D.V.Shirkov has considered the "heavy" fence example and has shown /7/ that the angle ζ satisfies the semi-group relation of type (11).

In the so-called main class of problems where forces and torques are concentrated at the rod ends $\sigma = 2\sqrt{P/H} = Const$ and the equation (32) admits the first integral (where κ^2 is integration constant):

$$\frac{d\zeta}{ds} = \sigma\sqrt{\kappa^2 - \sin^2\frac{\zeta}{2}}\,, \qquad (33)$$

having the solution

$$s = F(\Psi) - F(\Psi_0)\,, \qquad (34)$$

where

$$F(\Psi) = \frac{2}{\sigma}\int_0^{\Psi}\frac{d\psi}{\sqrt{1-\kappa^2\sin^2\psi}} \qquad (35)$$

and the following notations are introduced:

$$\Psi = \arcsin\left(\frac{1}{\kappa}\sin\frac{\zeta}{2}\right), \quad \Psi_0 = \Psi(\zeta_0)$$

The line shape is unambiguously determined by a function $\zeta(s,\zeta_0)$ (where ζ_0 is the tangent angle at a point $s=0$ defined by the semi-group relation

$$\zeta(s_1+s_2,\zeta_0) = \zeta(s_1,\zeta(s_2,\zeta_0)), \qquad (36)$$

which is completely similar to the relation (12). If we reduce the equation (36) to a differential form

$$d\zeta/ds = G(\zeta)$$

assuming s_2 small and defining $G(\zeta) = d\zeta/ds|_{s=0}$, then we shall obtain the equation (14) with generating function

$$G(\zeta) = \beta\sqrt{C - \sin^2\frac{\zeta}{2}} \qquad (37)$$

Now we note the analogy of equations (33) and (34) with (14) and (35), or with GellMannLow equation.

3.2.2 Example from ecology. Consider for simplicity a species of animals eating the same food, as well as devouring each other. If at initial time their number were Y_0 then let at a time t it be described by a function $Y(t,Y_0)$. The following semi-group property is obvious:

$$Y(t_1+t_2,Y_0) = Y(t_1,Y(t_2,Y_0))\,, \qquad (38)$$

which may be, in particular, reduced to a differential form used in ecology /18/. In such problems the function $G(u)$ is assumed to be quadratic.

By the way, the expressions (19) correspond to the particular type analogs of renormalization group equation solution in the double-charge model of QFT, and the particular stable solution of ecological problem about the beast-victim system.

In the discussed and similar mathematical modelling problems those advantages of semi-group description are important, of which we have spoken in section 1.4. Namely, instead of infinitesimal operator $G(u)$ one can determine the quantity $Y(\Delta,u)$ for arbitrary Δ, and then carry out the recurrent calculation by means of semi-group relation. The semi-group description also admits the corrections in the solution in the case when the latter diverges from the exact bahaviour prescribed by a semi-group property. And finally, one may use extensively the analog adoptions when solving the mathematical modelling problems.

3.3 Astrophysical Example.

Flare stars, from time to time, manifest short but relatively intensive flares. Statistical description of their ensemble is determined by means of stars number $n_k(T)$ which have shown exactly k flares during the entire observation time T. The problem stated by V.A.Ambartsumian/19/ is to find the true distribution of entire flare stars ensemble with respect to the flare frequency, which is actually a prediction problem for $n_k(t)$ for times $t>T$.

The vector $\vec{n}=\{n_k(t)\}$ should clearly satisfy the multiplicative version of non-linear semi-group relation (14)

$$\vec{n}\left(\frac{t}{T},\vec{n}(T)\right)=\vec{n}\left(\frac{t}{T_1},\vec{n}\left(\frac{T_1}{T},\vec{n}(T)\right)\right). \qquad (39)$$

Current number of flares may play the role of time parameter. In the particular case, when sequential flares for a single star and flares of different stars are independent (Poissonian property), this problem becomes linear and admits an explicit solution in form of "binomial expansion"/20/:

$$n_r(t)=\sum_{k=r}^{\infty} n_k(T)C_k^r\left(\frac{t}{T}\right)^r\left(1-\frac{t}{T}\right)^{k-r} \qquad (40)$$

where C_k^r is a number of combinations of r elements over k elements. Here $n_0(t)$ denotes the number of stars outburst during the period T but not detected by the time t. The probabilistic meaning of expression (40) is clear.

In the general case the quantities n_k may represent sums of any scallar quantities (both random or deterministic), which characterize the flare. For example, they may be equal to sum of amplitudes for stars with k flares, while this sum taken for all the number of flares may play the role of time parameter.

One may easily show that expression (40) describes both the past $t<T$ and the future of $n_k(t)$ for times $t>T$. Practically, due to the limited number of components for the observations vector $\{n_k(T)\}$, such prediction is possible only up to the times $t<2T$.

For a more accurate prediction it is desirable to have the $n_k(T)$ data free from the natural fluctuations. The best estimates on basis of observations data $n_k(t)$ obtained over the entire interval $(0,T)$ may be found by means of linear regression analysis, since the relations (40) are linear with respect to $n_k(T)$.

Observational data for Pleiades and Orion aggregates are in excellent agreement (by A.L.Mirzoyan and the author) with theoretical expression (40). Any deflections are caused by selective factors, when, for example, the observer detects the second flare of already known flare star with a higher probability than the first flare of the unknown star.

As already mentioned above, in this problem one may give other forms of linear semi-group relation over the increasing parameter - the number of flares:

$$n_{k+m}(t)=\int_0^t n_k(t-t')\,n_m(t')\,dt' \qquad (41)$$

and

$$n_k(t_1+t_2)=\sum_{m=0}^{k} n_{k-m}(t_1)\,n_m(t_2). \qquad (42)$$

The relation (42) is Kolmogorov-Chapman equation.

Note that the expression (40) corresponds to a system of differential equations

$$dn_k/dt=\alpha n_{k-1}-\alpha n_k \qquad (43)$$

whose solution is obtained with accuracy up to a set of arbitrary constants A_k, i.e. up to the arbitrary function $f(\nu)$ whose determination, being the subject of the considered problem, is made from the condition:

$$n_r(T)=\sum_{k=r}^{\infty}(-1)^r C_k^r A_k.$$

We also observe that if $n_k(T)$ is described by Poisson distribution or superposition of Poisson distributions, then the dependence of $n_r(t)$ on time for all t is presented by the same superposition of Poisson distribution.

Account of selectivities is the account of specific nonlinearities in process of flare detection. Such an account in its simplest form results in solution having the form of recurrent relations:

$$n_0(t) = Ne^{-\beta t}, \quad n_1(t) = \frac{\beta}{\alpha-\beta}\left[n_0(t) - Ne^{-\alpha t}\right],$$
$$n_{k+1}(t) = \frac{\alpha}{\alpha-\beta}\left[n_k(t) - N\frac{\beta}{\alpha}\frac{(\alpha t)^k}{k!}e^{-\alpha t}\right], \qquad (44)$$

where $\beta < \alpha$. When $\beta \to \alpha$ this distribution is transformed into Poisson distribution.

3.4 Other Astrophysical Problems

An increasing number of papers appear lately in the literature, in which a large part is given to the renorm-group approach in various astrophysical problems. In the transfer theory problems, as we have observed earlier, similar are relations based on invariance or invariant imbedding principles. One may attribute to these papers the pioneering researches of V.A.Ambartsumian on brightness fluctuation theory of the Milky Way, which attracted the principle of invariance to disclose the structure and properties of absorbing matter in the Galaxy.

The renorm-group approach in transfer theory has been applied recently in paper /21/, to modify the incoherent scattering analysis methods (in linear approximation) based on classical differential equation of transfer. In that paper the transformation has been used, which is reduced to scaling. The idea is that multiple walks whose frequencies are close to the centre of absorption line, may be described by the scattering process with large free path lengths for the quanta, similar to the one taking place at frequencies far from the line centre (in the line wings). Those operations are made

with a Fourier transforms of transfer equation and are very close mathematically to the well-known fast Fourier transform methods. Similar work in somewhat different aspect has been carried out in paper /22/ for the case of non-linear scattering. It should be mentioned, however, that our semi-group relations are absent in those papers in explicit form. Application of semi-group relations to the general transfer equation is given in paper /23/.

An attempt to use the semi-group relations in researches on stellar system dynamics was made in the Byurakan observatory. There are many principal difficulties originating here. To improve the efficiency of such approach many relations following from Ambartsumian's principle of invariance and other known principles of invariance with respect to the inertial frames of reference and mechanical similarity, were attracted by our colleague A.Beglarian. As a result, the system of obtained equations became complete. However, the analysis of such systems is possible today only for a small number of bodies and under very special initial conditions. Nevertheless, in a number of such cases we succeeded in finding unexpected solutions of the particular problems, which however may have applications in astrophysics. One advantage of such approach is that the numerical solution may be corrected by means of integral relations containing the information about all the behaviuor in the past.

We also point out the recently published paper /24/, stutying the mass distributionof expanding Universe. In that paper the renorm-group approach (actually it was a set of mechanical similarity relations, or scaling in time parameter) was used in order to minimize the numerical calculation errors, accumulated in the step-by-step solution of equations, which describe the cosmological model of expanding Universe.

All these papers show that analysis of group proper-
ies present in corresponding differential equations, hich describe various processes and phenomena in many ields of science, promises the obtaining of new research ethods.

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