NEW BALANCING PRINCIPLES APPLIED TO CIRCUMSOLIDS OF REVOLUTION, AND TO n-DIMENSIONAL SPHERES, CYLINDROIDS, AND CYLINDRICAL WEDGES

Tom M. Apostol and Mamikon A. Mnatsakanian

1 INTRODUCTION

The sphere and circumscribing cylinder engraved on Archimedes' tombstone commemorate his landmark discovery that their volumes and surface areas are related by the same constant ratio 2/3. He discovered the volume relation and many other geometrical results by mechanical balancing.

In particular, he found the volume of a cylindrical wedge by introducing the special balancing shown in Figures 1a and 1b. First, he balances lengths of horizontal chords of a triangle and a semicircular disk with respect to a vertical axis through its center as in Figure 1a. He then builds two-dimensional regions from these chords, the right triangle and semicircular disk in Figure 1b, which now are in area balance with each

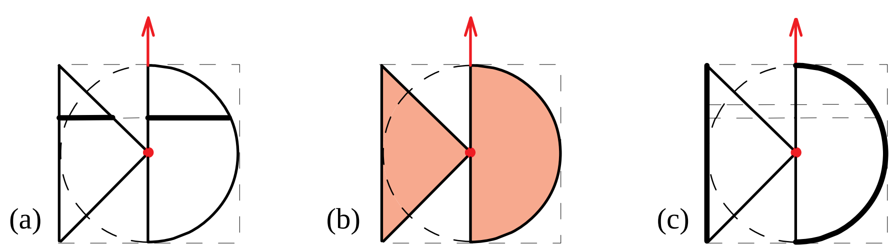

Figure 1: Archimedes' balancing of a triangle and semicircular disk: (a) chord-by-chord; (b) by areas. (c) Archimedes would have been pleased to learn that the semicirle and the vertical base of the triangle are in arclength equilibrium as well.

other. This balancing eventually led him to the volume of a cylindrical wedge. (See [5; Method, Prop. 11, p. 36], and Section 4 below for our treatment.)

Surprisingly, Archimedes' volume result on the sphere and cylinder can be directly deduced in two different ways from his own balancing in Figures 1a and 1b.

First, rotation of the chords in Figure 1a about the balancing axis, as indicated in Figure 2a, produces a circular annulus and a circular disk which, by a theorem of Pappus, have equal areas. Equality of these cross-sectional areas, in turn, yields equality of the volumes of the sphere and punctured cylinder (as was shown in [2]), giving Archimedes' 2/3 ratio for the sphere and its circumscribing cylinder.

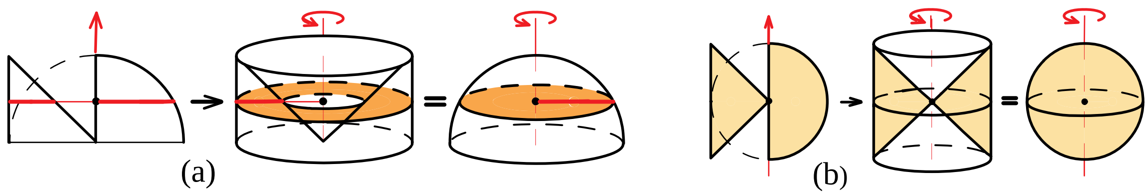

Figure 2: (a) Rotation of balanced chords produces cross sections of equal areas of a punctured cylinder and its insphere. (b) Rotation of balanced areas produces a punctured cylinder and its insphere of equal volumes.

Second, rotation of the two balanced areas in Figure 1b generates a punctured cylinder and the inscribed sphere as indicated in Figure 2b, which, by another theorem of Pappus, have equal volumes. This gives a second proof of Archimedes' volume result on the sphere and cylinder.

In [5; Method, p. 14] Archimedes makes the following comment: "*...I apprehend that some, either of my contemporaries or of my successors, will, by means of the method when once established, be able to discover other theorems in addition, which have not yet occurred to me.*"

This paper does that. It introduces new balancing principles and new ideas, including the concept of double equilibrium, that lead to many surprising results that Archimedes would have appreciated, all of which are extended to higher-dimensional space. One consequence of double equilibrium is a natural extension to n-space of Archimedes' results on both the volume *and* surface area of a sphere, in which the cylinder is replaced by a "cylindroid," and the fraction 2/3 is replaced by $2/n$ for all $n \geq 2$. The cylindroid is a cylinder only when $n = 3$.

Although the surface area relations for the sphere and cylinder do not follow from Archimedes' balancing in Figures 1a and 1b, they easily follow from our new balancing principles which imply, in particular, the arclength equilibrium in Figure 1c. This leads directly to a new proof of Archimedes' area result for the sphere and cylinder (see Figure 7b), and to a host of general surface area relations as well.

2 BALANCING REGULAR CIRCUMGONS IN A PLANE

Our methods are based on a new balancing lemma involving lengths of tangent segments to a circle, as depicted in Figure 3a. This simple lemma leads to profound consequences concerning equilibria of: arclengths of curves, areas of plane regions, as well as surface areas and volumes of various solids.

Main balancing lemma.

Figure 3a shows a fixed vertical balancing axis, indicated by the arrow, passing through the center of a circle of radius r. Consider an arbitrary tangent segment to the right of the axis of given length L with its midpoint on the circle, and its projection, of length H, onto a vertical tangent line on the other side of the axis.

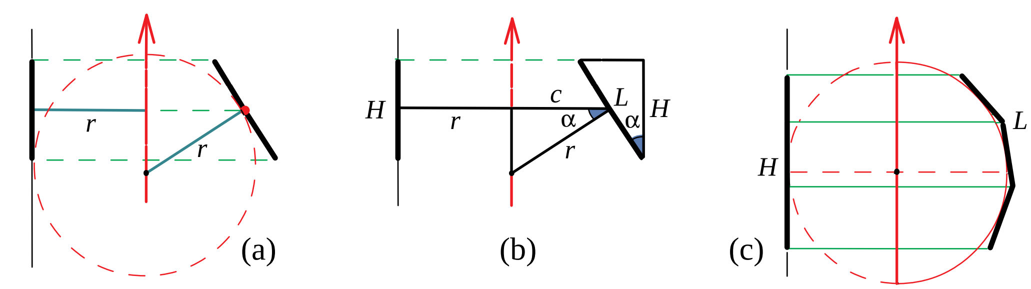

Figure 3: (a) Balancing a line segment and its projection. (b) Proof of the balancing principle. (c) Balancing a regular circumgon and its tangential projection.

If α is the angle between the tangent segment and the balancing axis, shown shaded in Figure 3b, we have $L \cos \alpha = H$. Multiplying by the radius r and introducing $c = r \cos \alpha$ we obtain the simple relation

$$Lc = Hr. \tag{1}$$

This equates moments of lengths L and H about the balancing axis, and establishes arclength equilibrium of the two segments: The length of an arbitrary tangent segment times the distance of its center from the balancing axis is equal to the length of its projection times its distance from the axis. We state this as follows:

Balancing Lemma 1. *A line segment tangent at its midpoint to a given circle is in arclength equilibrium, relative to an axis through the center of the circle, with its projection on a tangent line parallel to and on the other side of that axis.*

We turn next to extensions and applications of the balancing lemma.

Balancing regular circumgons with their tangential projections.

Circumgons are general objects circumscribing circles as introduced in [3]. This paper treats only regular circumgons, an example of which is shown in Figure 3c. It is composed of several connected segments of equal length tangent to the incircle and on the same side of a diameter. Their corresponding projected segments are also shown on the same vertical line to the left of the axis. Because the length of each edge of the circumgon is in equilibrium with that of its projection as indicated in (1), the entire circumgon is in arclength equilibrium with its entire projection, which we refer to as the *tangential projection*. In other words, (1) is still valid, where now L denotes the total length of the circumgon, c denotes the distance of its centroid from the balancing axis, and H denotes the total length of its tangential projection. The midpoint requirement in Lemma 1 restricts our circumgons to be regular, consisting of adjacent edges of equal length. Therefore we have:

Proposition 1. *A regular circumgon is in arclength equilibrium with its tangential projection.*

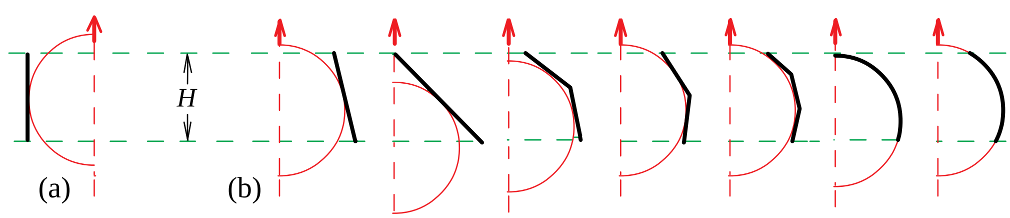

Figure 4: Balancing regular circumgons of a given height H in (b) with their tangential projections of total length H in (a). The circular arcs are limiting cases.

Figure 4b shows examples of regular circumgons of equal vertical height H in arclength equilibrium with a projected vertical segment in Figure 4a of the same height. All the circumgons have the same inradius but are located differently with respect to their common incircle. The last two examples are circular arcs, limiting cases of regular circumgons, whose arclengths are also in equilibrium with the same vertical projected segment of length H. Another example is the semicircular arc and its tangential projection in Figure 1c.

Balancing regular circumgonal regions with their tangential projection regions.

Figure 5 shows concentric regular circumgons in arclength equilibrium with their corresponding projections. By considering the union of an unlimited number of concentric circumgons in equilibrium with their corresponding projections, as suggested

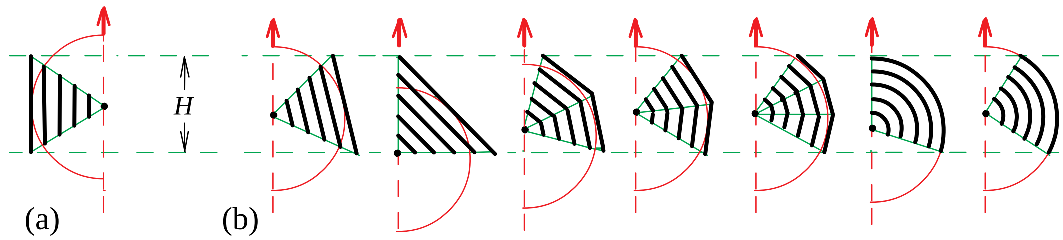

Figure 5: Balancing concentric regular circumgons in (b) with their respective projections in (a). The concentric circular arcs are limiting cases.

in Figure 6a, we can fill two-dimensional projection regions to also obtain areas in equilibrium as depicted by the examples in Figures 6b and c. Figure 6d shows the limiting case when the circumgons become circular arcs. This argument gives us:

Proposition 2. *A regular circumgonal region is in area equilibrium with its tangential projection region.*

The result can be written as an equation involving moments:

$$Ac_A = Pc_P, \tag{2}$$

where A is the area of the circumgonal region, P is the area of its projection region, and c_A, c_P are the respective centroidal distances from the balancing axis. This formula will be used later in Section 9.

In the special limiting case when the circumgonal region is a semicircular sector, the equilibrium in Figure 6d becomes Archimedes' area equilibrium in Figure 1b.

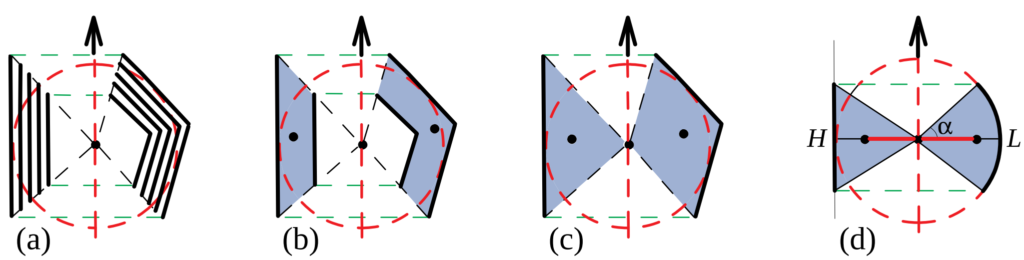

Figure 6: Circumgonal regions in area equilibrium with their corresponding tangential projection regions. The circular sector in (d) is a limiting case. The area equilibrium in (b) can also be verified directly by using known formulas for the centroid of a trapezoidal region.

Double equilibrium of regular circumgons.

Our process of building two-dimensional circumgonal regions as unions of concentric circumgons is somewhat analogous to Archimedes' process of building the two-dimensional triangle and semicircular disk in Figure 1b as unions of horizontal chords in Figure 1a. At each stage, balancing of lengths leads to corresponding balancing of areas. But our process also preserves balancing of circumgonal arclengths, including the limiting circular arcs, which Archimedes' process does not. Thus, Propositions 1 and 2 together give us *double equilibrium*, which can be described as follows:

Proposition 3. *A regular circumgonal region, including the limiting case of a circular sector, is in area equilibrium with its tangential projection region; in addition, its outer boundary is in arclength equilibrium with its tangential projection.*

3 THE BALANCE-REVOLUTION PRINCIPLE AND CIRCUMSOLIDS

This section introduces a balance-revolution principle that enables us to determine lateral surface areas and volumes of circumsolids of revolution by reducing them to those of cylinders.

Lateral surface areas.

Balance-revolution principle for surface areas. *If two plane curves are in arclength equilibrium with respect to a balancing axis, then the surfaces of revolution generated by rotating these curves about the balancing axis have equal areas.*

Arclength equilibrium means that corresponding moments are equal, and the balance-revolution principle follows by multiplying each member of this equality by 2π and applying Pappus' rule which, for surfaces of revolution, states that:

The area of a surface of revolution swept by a plane curve is equal to the length of the curve times the circumference of the circle traced by the centroid of the curve.

Figure 7a shows an example of a regular circumgon with two edges and its projection rotated together around their balancing axis. The rotated circumgon generates lateral surfaces of different truncated circular cones tangent to the insphere, and the vertical tangential projection generates a lateral cylindrical surface tangent to the insphere. By the balance-revolution principle, the area of the lateral surface of each truncated cone is equal to the lateral area of the corresponding *circumcylinder*. By additivity of area, the same is true for the composite surfaces. Although the circumgon in Figure 7a has only two edges, the same argument works for any regular

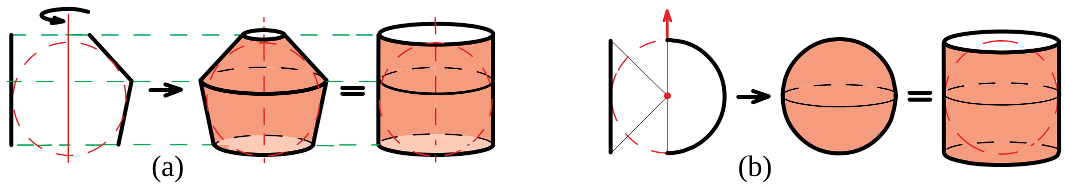

Figure 7: (a) Circumgon and its projection rotated about their balancing axis to form surfaces of equal area. (b) Semicircle and its projection in arclength equilibrium sweep out sphere and cylinder of equal area, a new proof of Archimedes' result.

circumgon. Figure 7b proves Archimedes' result that a sphere and its circumcylinder have equal areas because they are swept by rotating a semicircular arc and its projection about their balancing axis. These examples illustrate the following:

Theorem 1. *The lateral surface area of revolution generated by any regular circumgon, including the limiting case of a circular arc, is equal to that of the circumcylinder whose height is that of the circumgon.*

In other words,

$$S = 2\pi r H, \tag{3}$$

where S is the lateral surface area generated by any regular circumgon of revolution with inradius r and total height H in the direction parallel to the axis of rotation. This

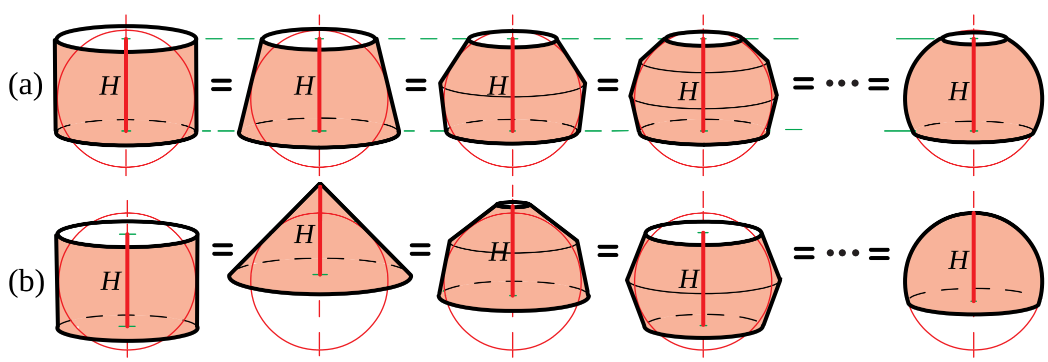

Figure 8: All regular circumgons of height H sweep out surfaces of area $2\pi r H$. All spherical zones of height H also have area $2\pi r H$.

striking result is exhibited in Figure 8, which shows surfaces of revolution generated by various circumgons of height H and the same inradius r. The same holds for the limiting case when the regular circumgon becomes a circular arc and its surface of revolution is part of a sphere. The area of a spherical zone of radius r lying between two parallel planes at distance H apart is $2\pi rH$, regardless of the location of the planes. Figure 8b shows circumgons located differently with respect to the diameter of the incircle. In particular, the surface area of a cone, truncated or not, is equal to that of a cylinder of the same height and the same insphere. This is a simple and elegant way of relating the cone to the cylinder.

Consider the special case of Theorem 1 in which the circumgon is half a regular $2n$-gon rotated around a diameter of the incircle perpendicular to two opposite sides of the circumgon, as shown by the examples in Figure 9. We call these *right* circumgonal surfaces of revolution. They all have height $2r$. The circumscribing cylinder is the case $n = 2$. All these lateral surfaces have equal area $4\pi r^2$, which is also the area of

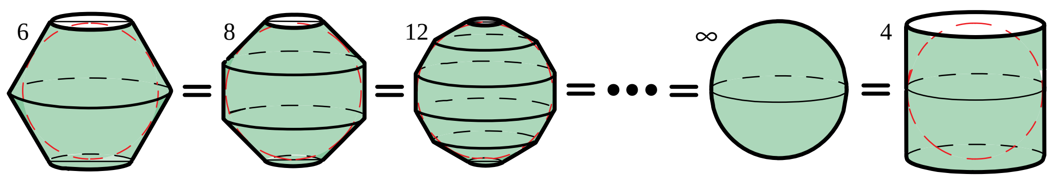

Figure 9: All right circumgonal surfaces of revolution of height $2r$, including the insphere, have equal lateral area, which is that of the circumcylinder, $4\pi r^2$.

the limiting sphere. This gives us the following remarkable extension of Archimedes' area result on the sphere and cylinder:

Corollary of Theorem 1. *All right circumgonal surfaces of revolution with inradius r have height $2r$ and lateral surface area $4\pi r^2$.*

Volumes.

Figure 10b shows two circumgonal regions in area equilibrium obtained by balancing, as was done earlier in Figure 6c, a collection of concentric circumgons and their corresponding projections in Figure 10a. When these regions are rotated together about the balancing axis they sweep out two solids of revolution having equal volumes because of the following principle, illustrated in Figure 11.

Balance-revolution principle for volumes. *If the areas of two plane regions are in equilibrium with respect to a balancing axis, then the solids of revolution generated by rotating these regions about the balancing axis have equal volumes.*

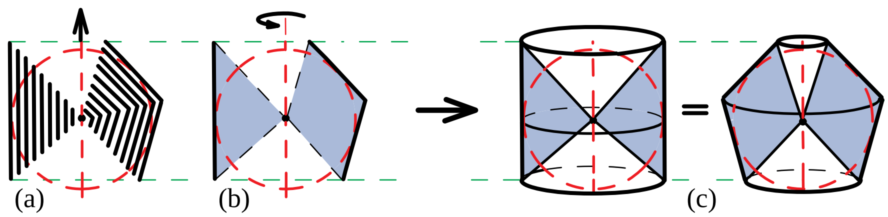

Figure 10: Rotating two circumgonal regions in area equilibrium in (b) gives two punctured solids of revolution in (c) with equal volumes.

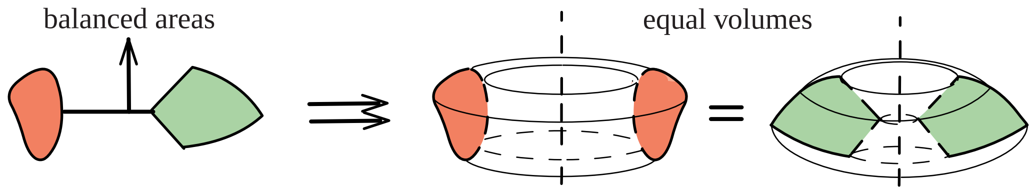

Figure 11: Rotating two plane regions in area equilibrium about the balancing axis produces solids of revolution having equal volumes.

The balance-revolution principle for volumes follows from Pappus' rule for volumes, which is analogous to that for surfaces, applied to solids of revolution.

The two solids shown in Figure 10c have equal volumes. They are obtained by rotating the triangle and quadrilateral in area equilibrium in Figure 10b. One is a solid circular cylinder punctured by two cones, and the other consists of two parts, each being a frustum of a solid cone punctured by another cone.

Figure 12 shows further examples illustrating the balance-revolution principle. Each pair of solids is generated by rotating about the balancing axis two circumgonal regions in area equilibrium, so they have equal volumes.

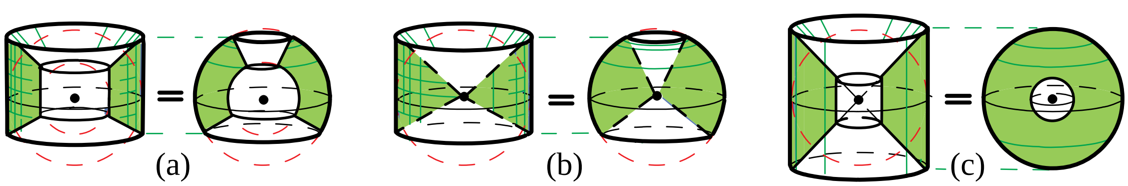

Figure 12: Circumsolids of revolution. In each pair, heights and volumes are equal.

Figure 13 shows solids of revolution generated by regular circumgonal plane regions and their tangential projections, all of altitude H. Each solid is punctured by two

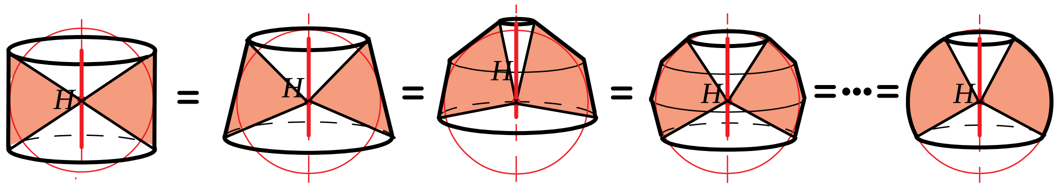

Figure 13: Regular circumgonal regions of equal altitude in area equilibrium generate punctured solids of revolution of equal volume.

cones whose vertices are at the center of the insphere. Because they circumscribe a sphere, all these are examples of *circumsolids*, as described in [4], and they illustrate the following general theorem:

Theorem 2. *The volume of any punctured circumsolid generated by rotating a regular circumgonal region is equal to that of the punctured circumcylinder whose height is that of the circumgon.*

Each such solid has volume V of a punctured cylinder given by

$$V = \frac{2}{3}\pi r^2 H. \tag{4}$$

This result is consistent with our knowledge of circumsolids. Each solid of revolution is a circumsolid so, by Theorem 1 in [4], each volume is one-third the product of its outer surface area, $2\pi r H$, and the radius of the insphere, in agreement with (4).

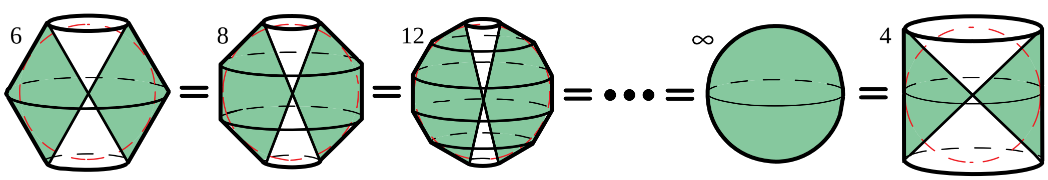

Figure 14: All punctured right circumgonal solids of revolution of height $2r$ have the same volume as the insphere and its punctured circumscribing cylinder, $4\pi r^3/3$.

Figure 14 shows examples of *right circumsolids*, obtained by rotating half a regular $2n$-gonal region of the type shown in Figure 9. For such circumsolids we have the following extension of Archimedes' volume result on a sphere and its punctured circumscribing cylinder, which is the case $n = 2$:

Corollary of Theorem 2. *All punctured right circumsolids of revolution with inradius r have height $2r$ and volume $4\pi r^3/3$.*

4 MOMENT-WEDGE PRINCIPLE AND CYLINDRICAL WEDGES

As noted, Archimedes used area balance in Figure 1b to find the volume of a cylindrical wedge, the main new result of his Method. Using a new principle, we shall obtain volumes of a family of wedges with circumgonal bases, which includes the Archimedes wedge as a limiting case. A similar treatment is given for their lateral surface areas. Incidentally, Archimedes did not treat the surface area of a cylindrical wedge.

Moment-wedge volume principle and circumgonal wedges.

This principle, stated in (5), relates the area moment of the planar base of a right cylinder, with respect to an axis in the plane of the base, with the volume of a wedge cut from that cylinder by an inclined plane. Figure 15a shows a right cylinder with a horizontal planar base of general shape cut by a plane whose angle of inclination with the horizontal has tangent k. We are interested in the area moment of the base relative to the line of intersection of the two planes, which we take as an axis of moments. An element of area $\Delta x \Delta y$ at distance x from the axis has moment $\Delta M = x \Delta x \Delta y$. On the other hand, the cylindrical column shown is an element of volume given by $\Delta W = kx \Delta x \Delta y$. Therefore,

$$\Delta W = k \Delta M.$$

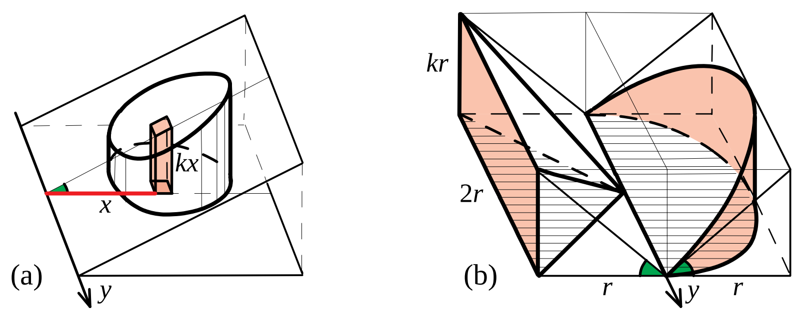

Figure 15: (a) Relating the volume of a general cylindrical wedge to the area moment of its base. (b) Volume of Archimedes' cylindrical wedge is equal to that of the pyramid because their bases are in area equilibrium.

Integration gives

$$W = kM, \tag{5}$$

where W is the volume of the cylindrical wedge whose base has area moment M about the horizontal y axis.

In Figure 15b we apply (5) to two cylindrical wedges whose horizontal bases are the triangle and semicircular disk of Figure 1b. As we know, these two bases are in area equilibrium about the y axis, so moment M is the same for the two bases if the two cutting planes are inclined at the same angle with the horizontal. Hence, by (5), the two wedges have equal volumes $W = kM$. Thus, the volume of the cylindrical wedge is equal to that of the pyramid whose rectangular base is shaded in Figure 15b. This gives $W = 2kr^3/3$, one-third the area of the base times the altitude. Archimedes deduced (by a completely different method) the equivalent result that the volume of the cylindrical wedge is one-sixth that of the largest rectangular box in Figure 15b.

The same idea applied to any two general cylindrical wedges whose bases are in area balance gives the following corollary:

Balance-wedge volume principle. *Two cylindrical wedges have equal volumes if their bases are in area equilibrium about the line of intersection of the two cutting planes inclined at the same angle with the common base plane.*

Right cylinders include prisms with circumgonal bases. We apply the foregoing corollary to prisms with circumgonal bases in area balance. Those in Figure 16 are built from halves of regular $2n$-gons in area balance with the same triangular base. All these wedges have volume equal to that of the pyramid with rectangular base.

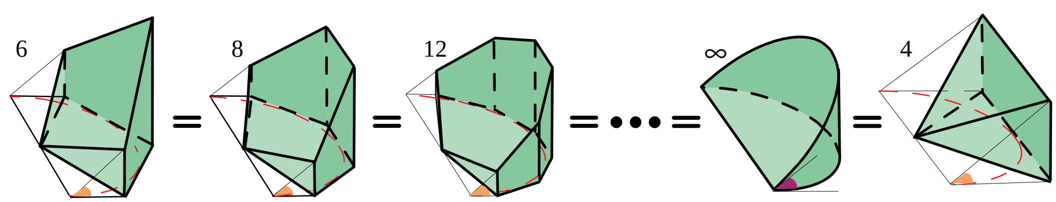

Figure 16: Circumgonal wedges of equal volume. Their bases are halves of regular $2n$-gons, all in area balance. Archimedes' wedge in Figure 15b is their limiting case.

Lateral surface area of a circumgonal wedge.

If the circle in Figure 15b has radius r, the shaded rectangle has base $2r$, altitude kr, and area $2r^2k$. This is also the lateral surface area of the cylindrical wedge, shaded in Figure 15b. To verify this we note that by double equilibrium (Proposition 3), the semicircle and its horizontal projection in the plane of the base are in arclength equilibrium. Thus an element of arc of length Δs and its projection Δp have equal moments about the diameter, $\Delta s \cdot x = \Delta p \cdot r$, where x is the distance of element Δs from the diameter. The corresponding elements of lateral surface area are $\Delta s \cdot kx$

for the cylinder and $\Delta p \cdot kr$ for the projection, which are equal because the moments are equal. The sum of all the Δp is $2r$ so by integration with respect to arclength we see that the lateral surface of the cylindrical wedge is $2r^2k$, the same as that of the projection rectangle. The same applies to the lateral surface of each wedge in Figure 16, which consists of trapezoidal faces whose total area is equal to that of the common projection rectangle. A similar proof works for a general cylindrical wedge like that in Figure 15a whose planar base is bounded by a rectifiable curve.

Thus, we have shown that the volume W of a cylindrical wedge and its lateral surface area A are given by the following formulas:

$$W = \frac{2}{3}kr^3, \ A = 2kr^2.$$

The results in this section will be extended to higher dimensions in Section 8.

Figure 17 shows examples of those lateral surfaces in Figure 16 with $r = k = 1$. Only half the unwrapped lateral surface is shown, the other half being symmetric. Each unwrapped surface shown in Figure 17 has area 1, which is the area of the common projection unit square. In the limiting case when the base is circular, the unwrapped portion of the cylindrical surface is the region under half a sine curve.

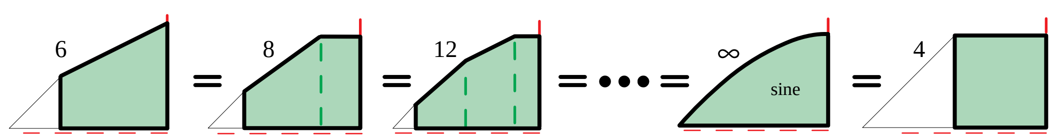

Figure 17: Halves of unwrapped lateral surfaces of the circumgonal wedges in Figure 16. All have the area of a unit square, the same as the region under half a sine curve.

5 BALANCING PORTIONS OF A SPHERE AND OF A CYLINDER

This section extends Balancing Lemma 1 by establishing equilibria of various portions of a sphere and of a circular cylinder with their tangential projections.

Surfaces on a sphere.

We can generate surfaces on a sphere by rotating a circular arc and its projection about a horizontal axis through the center of the circle, as indicated in Figure 18a. Let Δs denote the length of the small circular arc in Figure 18a. Then its projection has length Δp which satisfies the approximate relation $\Delta s \cdot \cos\alpha = \Delta p$, where α is the angle of inclination shown. Multiplying by the radius r to obtain moments about the balancing line, we find

$$\Delta s \cdot (r \cos \alpha) = \Delta p \cdot r,$$

which can be regarded as an approximate balancing relation. Rotating the balancing line generates a central balancing plane, and rotating the projection line generates a projection plane, both perpendicular to the axis of rotation, as in Figure 18b. The

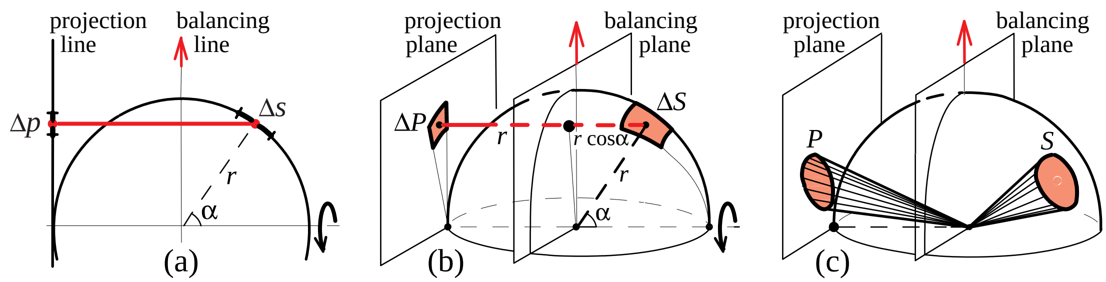

Figure 18: Rotating a circular arc and its vertical projection in (a) about a horizontal axis generates a surface element in (b) in area equilibrium with its projection relative to a balancing plane. (c) Solid angle in double equilibrium with its projection cone.

circular arc generates an element of spherical surface area ΔS obtained by rotation through a small angle. Let ΔP denote the area of the projection of ΔS on the tangent projection plane. They are in the approximate relation $\Delta S \cdot \cos \alpha = \Delta P$. Multiplying by r to obtain moments, we find

$$\Delta S \cdot (r \cos \alpha) = \Delta P \cdot r, \tag{6}$$

which is an approximate area balancing relation. The factor $r \cos \alpha$ is the centroidal distance of a small spherical surface area element from the balancing plane, r is the centroidal distance of the projection from the balancing plane, and (6) represents area equilibrium of elements ΔS and ΔP with respect to this plane.

In particular, each small area element in Figure 18b will be in area equilibrium with its corresponding planar projection according to (6). Because any region of area S on the sphere and its projection of area P on the projection plane can be approximated arbitrarily closely by such area elements, (6) gives us the following area equilibrium relation:

$$Sc = Pr, \tag{7}$$

where r is the radius of the sphere and c is the centroidal distance of the surface of area S from the balancing plane. This relation, illustrated in Figure 19, states that:

Balancing Lemma 2. *Any portion of a spherical surface is in area equilibrium with its tangential projection relative to a balancing plane through the center of the sphere.*

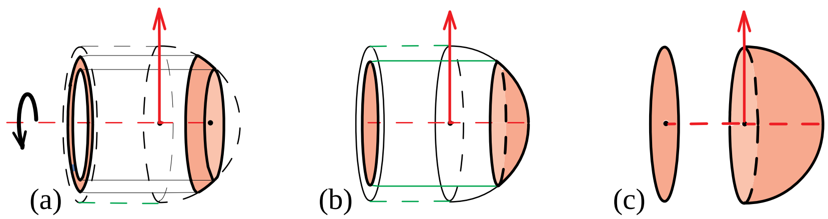

Figure 19: Balancing: (a) spherical zone and its projection annulus; (b) spherical cap and its projection disk; (c) hemispherical surface and its projection disk.

Double equilibrium of a solid angle and its projection cone.

Figure 18c shows a solid angle of radius r, a union of radial lines emanating from the center of the sphere to a portion of the sphere with area S. It is also a union of concentric layers of spherical surfaces formed by radial shrinking of the spherical surface of area S to 0. The spherical surface of each layer is in area equilibrium with its projection on a plane tangent to the concentric sphere. By multiplying (6) by the thickness Δr of a typical layer, we obtain layer by layer volume equilibrium. Consequently, the entire solid angle is in volume equilibrium with its projection solid, a cone whose vertex is at the center of the sphere and whose base is the projection of the region of area S. The solid spherical sector in Figure 20a is a special case. It can

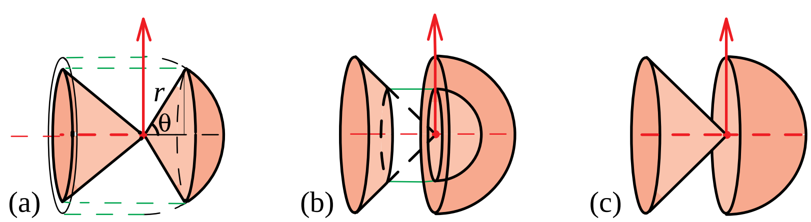

Figure 20: Double equilibrium of: (a) spherical sector and cone; (b) hemisphere with cavity and truncated cone; (c) hemisphere and solid cone.

also be formed from a collection of concentric hemispherical caps in area equilibrium with corresponding projected disks as in Figure 19b to produce a spherical sector in volume equilibrium with its projection cone.

Similarly, we can use concentric hemispherical surfaces like those in Figure 19c to build a solid hemisphere in Figure 20c in volume equilibrium with a solid cone built from the corresponding projected equatorial disks. Figure 20b shows an intermediate stage, a solid hemisphere with a cavity in equilibrium with a truncated cone.

Balancing axes for symmetric objects.

The equilibria in Figures 19 and 20 are with respect to a balancing plane. All these objects were obtained by rotation about a horizontal axis, which is also an axis of symmetry that intersects the balancing plane at the centroid of each composite object. Consequently, any axis through this centroid will also be a balancing axis. This also applies to more general situations when the centroid of a composite object is determined by symmetry, as in Figures 1, 19, 20, 21a, 22c, 23, 24, 25, 27a and 27c.

Double equilibrium: spherical wedge and elliptical cone; punctured spherical zone and projection cone.

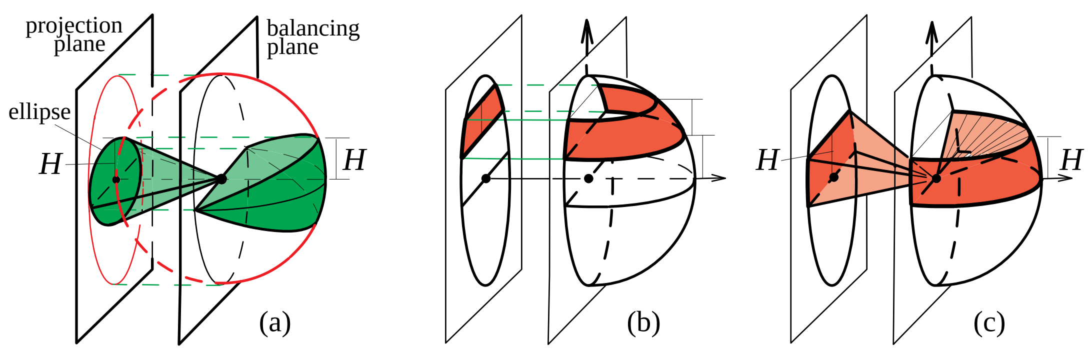

Figure 21: (a) Double equilibrium: spherical wedge and cone. (b) Area equilibrium: zone and its projection. (c) Double equilibrium: punctured spherical zone and cone.

Figure 21a shows a spherical wedge of height $2H$ cut from a solid hemisphere of radius r by two planes through a diameter, each making an angle θ with the equatorial plane. The spherical surface of the wedge is in area equilibrium with its projection, which is an ellipse whose major axis has length $2r$ and whose minor axis has length $H = 2r \sin\theta$. The solid spherical wedge is also in volume equilibrium with the solid elliptical cone. Figure 21b shows a spherical surface of a zone in area equilibrium with a slice of a circular disk, and Figure 21c shows double equilibrium of solids built from concentric figures of the type in Figure 21b.

Balancing portions of a circular cylinder.

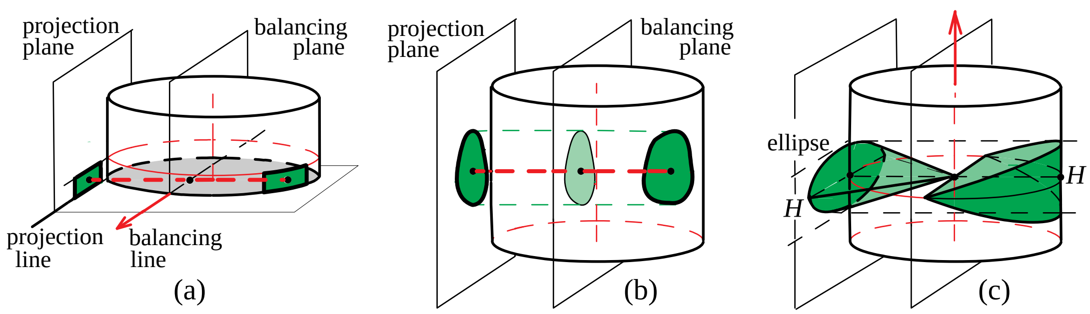

Figure 22: (a) Small area element on a cylinder in area balance with its rectangular projection. (b) Region on a cylinder in area balance with its tangential projection. (c) Cylindrical wedge in double equilibrium with its projection elliptic cone.

Instead of rotating the circle in Figure 18a to generate a sphere, we can translate it in a direction perpendicular to its plane to generate a circular cylinder of radius r, as suggested by Figure 22a. Corresponding translations of the balancing line and projection line in Figure 18a generate two parallel planes which we call the balancing plane (a bisector of the cylinder) and the projection plane (tangent to the cylinder), as shown in Figure 22a. Translation of the circular arc in Figure 18a sweeps out an area element (on the lateral surface of the cylinder) in area balance with its vertical projection rectangle (on the projection plane), as indicated in Figure 22a. More generally, any region of area S on the cylinder will be in area equilibrium, with respect to the balancing plane, with its corresponding projection of area P on the projection plane, as illustrated in Figure 22b, because they can be approximated arbitrarily closely by area elements like those in Figure 22a. Again, (7) holds, where now S is the area of the cylindrical region, P is the area of its projection and c is the centroidal distance from the balancing plane to the cylindrical surface of area S. This gives us:

Balancing Lemma 3. *Any portion of a cylindrical surface is in area equilibrium with its tangential projection relative to a balancing plane through the axis of the cylinder.*

Figure 22c shows a solid cylindrical wedge of height H between two inclined planes, each making an angle θ with the horizontal equatorial plane. The lateral surface of the wedge is in area equilibrium with its projection, which is an ellipse whose major axis has length $2r$ and whose minor axis has length H. By using concentric cylindrical wedges of decreasing radii, we can build a solid cylindrical wedge in double equilibrium with a solid elliptic cone as shown in Figure 22c. Not only is the lateral cylindrical surface in area equilibrium with its elliptical projection, but also the solid cylindrical wedge is in volume equilibrium with the elliptic cone.

The spherical wedge in Figure 21a and the cylindrical wedge in Figure 22c are in double equilibrium with the same elliptic cone. Eliminating the cone, we obtain the double equilibrium in Figure 23, which can be described as follows:

Proposition 2. *A spherical wedge is in double equilibrium with a cylindrical wedge of the same height and the same insphere.*

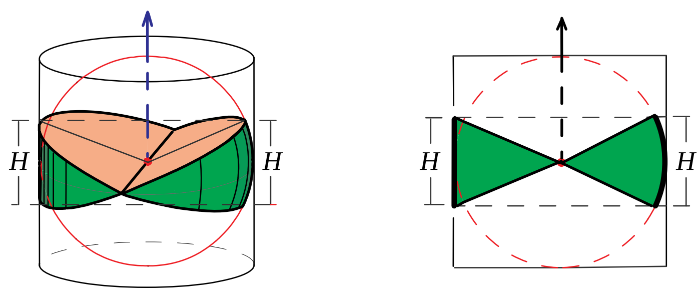

Figure 23: Spherical and cylindrical wedges in double equilibrium.

When the wedges in Figure 23 are viewed along the common diameter, we see the area and arclength equilibrium obtained earlier in Figure 6d.

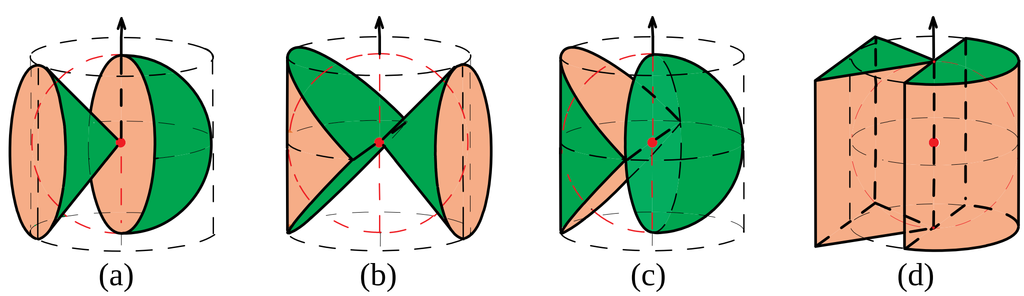

Figure 24: Solids in double equilibrium. The top view of (d) appears in Figure 1b and also represents the top view of each of (a) and (b) and the side view of (c).

Figure 24a shows Figure 21a when the wedge is a hemisphere, and Figure 24b shows the cylindrical wedge and cone in Figure 22c when the altitude is a diameter. The equilibrium in Figure 24c is obtained by eliminating the cone in the previous two figures. Figure 24d shows a triangular prism and semicircular cylinder also in double equilibrium. Archimedes established their volume equilibrium, which is equivalent to area equilibrium of the triangle and semicircular disk in the top view.

The balancing relations just established, although of interest in their own right, will be extended in the next section and applied to obtain equality of volumes and surface areas of higher-dimensional solids.

6 HIGHER-DIMENSIONAL BALANCING PRINCIPLES

We begin this section by analogy with the process illustrated in Figure 2b, where the balanced 2-d regions were rotated in 3-space about the balancing axis to generate a solid sphere and punctured cylinder. Now we do the same with the cone and hemisphere in equilibrium in Figure 24a. Imagine these solids imbedded in 4-space and rotate each around the balancing axis to sweep out 4-dimensional solids. The hemisphere generates a 4-dimensional solid sphere, and the cone generates an object which we call a *punctured* 4*-cylindroid*. To find relations between their volumes and surface areas, we need a higher-dimensional balance-revolution principle.

Double equilibrium of solid angle with its projection cone.

We will extend some of our earlier balancing relations to higher-dimensional space in a manner completely analogous to the transition from 2-d in Figure 18a to 3-d in Figure 18b. First, regard the 3-d configuration in Figure 18b as imbedded in 4-space, and rotate the configuration in 4-space around the horizontal axis of symmetry of the hemisphere. The 3-dimensional hemisphere to the right of the balancing plane produces a 4-dimensional hemisphere, the balancing plane produces a balancing hyperplane, and the projection plane produces a projection hyperplane in 4-space. If rotated through a small angle, the 3-d surface element produces a corresponding surface element on the 4-sphere, and the surface element on the projection plane produces a surface element of the projection hyperplane in 4-space, which is the projection of the surface element on the 4-sphere.

The same type of argument that produced the balancing relations (6) and (7) shows that two elements in 4-space satisfy the same kind of balancing relation with respect to the balancing hyperplane.

By repeating the process and arguing by induction on the dimensionality of the space, we find that the same type of balancing holds in n-space, because the angle of inclination α of the radial line being rotated in Figures 18a and 18b is the same for all dimensions. This leads to double equilibrium, with respect to the balancing plane, of an n-dimensional solid angle and its projection n-cone built from parallel projection bases, by analogy with the solid angle equilibrium described in Figure 18c.

Solid angle balancing lemma. *An n-dimensional solid angle and its projection n-cone are in double equilibrium with respect to a balancing hyperplane that passes through the vertex of the cone and is parallel to the projection hyperplane.*

Moment-volume principle.

The following principle relates moments and "volumes" of revolution. We place quotation marks around the word "volume" to state one principle that applies to both surface areas and volumes as understood in the traditional use of the terms.

Moment-volume principle. *When an object in n-space with moment M_n relative to an axis is rotated in $(n+1)$-space about that axis, it produces an object whose "volume" V_{n+1} is given by*

$$V_{n+1} = 2\pi M_n. \tag{8}$$

In other words, the volume of an $(n+1)$-dimensional solid of revolution is simply 2π times the n-dimensional moment relative to the axis of rotation of the object that generates this solid.

The idea of the proof can be easily seen when $n = 2$. Imagine an area element $\Delta x \Delta y$ at distance x from the y-axis in the xy plane. Its moment about the y axis is $\Delta M_2 = x\Delta x \Delta y$. When this is rotated about the y-axis it sweeps out a solid of volume $\Delta V_3 = 2\pi x \Delta x \Delta y = 2\pi \Delta M_2$. Similarly, for higher dimensions we find

$$\Delta V_{n+1} = 2\pi \Delta M_n.$$

Integration then gives (8).

In particular, we have the following corollary:

Balance-revolution principle. *If two objects in n-space are in "volume" equilibrium with respect to a balancing axis, then the objects in $(n+1)$-space generated by rotating them about this axis have equal "volumes."*

In the balance-revolution principle for surface areas stated at the beginning of Section 3, the objects in 2-space in "volume" equilibrium are two plane curves in arclength equilibrium. When they are rotated about the balancing axis they generate surfaces in 3-space with equal "volumes", which now means equal surface areas. In the balance-revolution principle for volumes stated later in Section 3, the objects in 2-space in "volume" equilibrium are two plane regions in area equilibrium. When they are rotated about the balancing axis they generate solids in 3-space with equal "volumes", which now means equal volumes in the traditional sense.

Applications to n-hemispheres.

The following specialization of the solid angle balancing lemma plays a key role in the sequel:

Special balancing lemma. *An n-hemisphere is in double equilibrium with its projection n-cone with respect to the vertex of the cone and to any axis through it.*

This special lemma, combined with the balance-revolution principle, leads to:

Double equality principle. *The objects obtained by rotating this n-hemisphere and n-cone in $(n+1)$-space have equal lateral surface areas and equal volumes.*

7 ON THE SPHERE AND CYLINDROID IN n-SPACE

The main result of this section is an extension of Archimedes' classical results on volumes and surface areas of spheres to n-space for all $n \geq 2$, stated below in Theorem 3. This is perhaps the most profound consequence of our balancing methods because when $n \neq 3$ no simple relation connects the volume or the surface area of an n-sphere with that of its circumscribing n-cylinder. (See [6] and [7] for attempts to find such a relation.) In our extension process a new object occurs naturally, an n-cylindroid, which has a simple and direct relation to its insphere. For $n = 3$ a cylindroid is a traditional cylinder, but when $n \neq 3$ it is an entirely different object.

Natural evolution of the cylindroid.

Figures 25 and 26 illustrate how our balancing methods lead naturally to the concept of cylindroid. In Figure 25, at the top of the first column, the semicircular disk and triangle are in double equilibrium about a vertical axis through a diameter. When these are rotated in 3-space about the balancing axis they produce two solids of revolution, the sphere and punctured cylinder shown below them, which, by the double equality principle, have equal volumes and equal lateral surface areas. This is Archimedes' classical result for $n = 3$, which we deduced by using double equilibrium.

When the triangle and semicircular disk in the first row are rotated in 3-space about the *horizontal* axis of symmetry they produce a solid hemisphere and cone, in double equilibrium about the same vertical axis, shown at the top of the second column in Figure 25. Rotating these solids, in turn, in 4-space about the balancing axis produces two 4-dimensional solids of revolution shown at the bottom of the second column, which, by the balance-revolution principle, have equal "volumes," which here means both 4-dimensional volumes and lateral surface areas are equal.

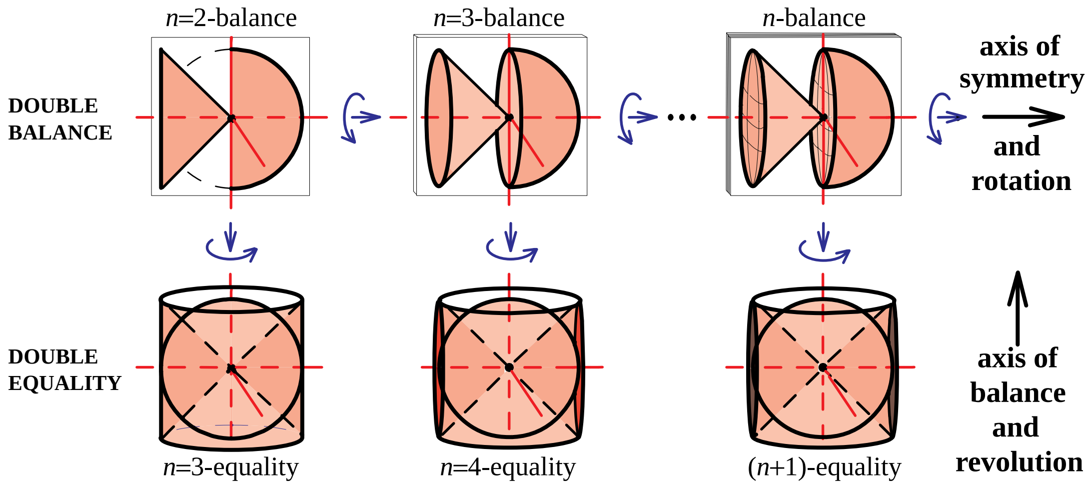

Figure 25: Diagram illustrating the natural evolution of the cylindroid.

To help visualize the process, refer to Figure 26a, showing a solid cone inscribed in a circular cylinder in 3-space. Figure 26b shows a schematic "flattening" of 3-space into a hyperplane in which the cone and cylinder appear as a triangle inscribed in a rectangle. We choose a 4th coordinate axis perpendicular to this hyperplane (labeled as *next* in Figure 26b) then rotate the hyperplane with its contents around an axis that is perpendicular to this 4th axis and also to the axis of symmetry of the cone and cylinder. If we would rotate about the axis of symmetry, we would obtain a 4-cylinder with an inscribed 4-cone. Instead, we obtain new objects, a 4-*cylindroid*, and the subset swept by the cone, a *punctured* 4-*cylindroid*, depicted in Figure 26c. The flattened base of the triangle in Figure 26b is actually the circular base of the cone in Figure 26a. Rotating this circular base generates the lateral surface of the cylindroid in Figure 26c.

We continue the process in Figure 25, successively rotating the double-balanced objects in the top row through one higher dimension around the horizontal axis of symmetry to obtain a higher-dimensional hemisphere and cone in double equilibrium about the same vertical balancing axis through the vertex of the cone. When these balanced symmetric objects are rotated through one higher dimension around the balancing axis they produce two solids of revolution of equal "volumes," a higher-dimensional sphere and a punctured higher-dimensional cylindroid, the key that enables us to extend Archimedes' results to n-space for every value of $n \geq 2$.

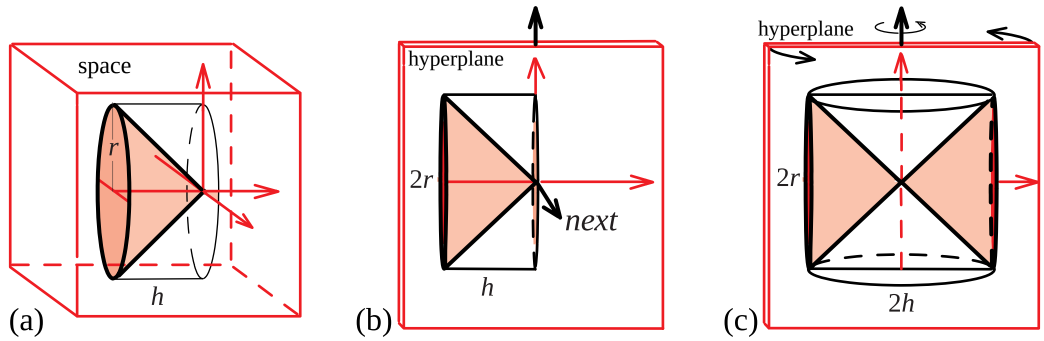

Figure 26: Constructing a 4-dimensional cylindroid and a punctured cylindroid. (a) Solid cylinder and inscribed cone in 3-space. (b) Schematic flattening of (a) onto a hyperplane, with a 4th axis erected perpendicular to it. (c) Rotation of (b) in 4-space around a coordinate axis perpendicular to this 4th axis and to the axis of symmetry.

Definitions of cylindroid and punctured cylindroid.

In general, an n-cylindroid and a punctured n-cylindroid are defined by direct analogy with the 4-dimensional case. We begin with an $(n-1)$-cylinder of radius r and altitude h and the corresponding inscribed isosceles $(n-1)$-cone by analogy with Figure 26a. Rotate them together in n-space around a coordinate axis that passes through the vertex of the cone and is also perpendicular to the axis of the cylinder. During the rotation, the $(n-1)$-cylinder sweeeps out an object we call an *n-cylindroid* of radius h and altitude $2r$. As the $(n-1)$-cylinder sweeps out the cylindroid, the $(n-1)$-cone sweeps out a portion of the n-cylindroid that we call a *punctured cylindroid*.

The common base of the $(n-1)$-cylinder and cone is an $(n-2)$-sphere of radius r. Cross-sections of the $(n-1)$-cylinder perpendicular to its axis are $(n-2)$-spheres of constant radius r, whereas cross-sections of the $(n-1)$-cone are $(n-2)$-spheres whose radii decrease linearly from r at the base to 0 at the vertex.

When $n \neq 3$, an n-cylindroid differs from an n-cylinder. In particular, when $n = 2$, a 2-dimensional cylinder is a rectangle, but a 2-dimensional cylindroid is a circular disk, obtained by rotating a 1-cylinder (a line segment) about one of its endpoints. Moreover, the "cone" inscribed in that 1-cylinder is the 1-cylinder itself so the punctured and unpunctured 2-cylindroid are the same object.

Moment relations for cone and cylinder.

The following formulas for the volume $v_{n-1}(\text{cyl})$ of the $(n-1)$-cylinder and the volume $v_{n-1}(\text{cone})$ of an $(n-1)$-cone are known:

$$v_{n-1}(\text{cyl}) = hV_{n-2}, \quad v_{n-1}(\text{cone}) = \frac{v_{n-1}(\text{cyl})}{n-1} = \frac{h}{n-1}V_{n-2},$$

where V_{n-2} is the volume of their common $(n-2)$-spherical base, and h is their common altitude. It is also known that the centroid of the $(n-1)$-cylinder is at the mid-point of its altitude, while that of the $(n-1)$-cone is at a distance h/n from the base or, equivalently, at distance $h(n-1)/n$ from the vertex. Consequently, the volume moments of the cylinder and cone with respect to any axis through its vertex perpendicular to its axis of symmetry are given by

$$M_{n-1}(\text{cyl}) = \frac{h^2}{2}V_{n-2}, \quad M_{n-1}(\text{cone}) = \frac{h^2}{n}V_{n-2}. \tag{9}$$

From (9) we immediately obtain the following lemma which plays an important role in our extension of Archimedes' results.

Moment ratio lemma. *The volume moment of an $(n-1)$-cone with respect to any axis through its vertex is $2/n$ times the corresponding moment of its circumscribing $(n-1)$-cylinder; hence the ratio of the two moments, cone to cylinder, is $2/n$.*

Extension of Archimedes' classical results to n-space (suitable for engraving on Archimedes' hypertombstone).

Now we can state and prove the main result of this section, valid for all $n \geq 2$:

Theorem 3. (a) *The volume of an n-sphere equals that of its punctured circumscribing n-cylindroid.*

(b) *The surface area of an n-sphere is equal to the lateral surface area of its circumscribing n-cylindroid.*

(c) *The volume of an n-sphere is $2/n$ times that of the volume of its (unpunctured) circumscribing n-cylindroid.*

(d) *The surface area of an n-sphere is $2/n$ times that of the total surface area of its (unpunctured) circumscribing n-cylindroid.*

Proof of 3a *and* 3b: These follow directly from the double equality principle stated at the end of Section 6.

Proof of 3c *and* 3d: By the moment ratio lemma, the volume moments of the $(n-1)$-cone and $(n-1)$-cylinder about any axis through the vertex of the cone have ratio $2/n$. By the moment-volume principle (8), (with n replaced by $n-1$), the punctured and unpunctured cylindroids in n-space generated by rotating the cone and cylinder have the same volume ratio $2/n$. Combining this with Theorem 3a and 3b we obtain Theorem 3c and 3d.

Actually, Theorems 3a and 3b are equivalent because both the n-cylindroid and the punctured n-cylindroid are circumsolids, so by [4; p. 540] their volumes have the same ratio as their surface areas. The same remark applies to Theorems 3c and 3d.

Note that the 2-cylindroid and the punctured 2-cylindroid are the same object, which is identical to their insphere (a disk), consistent with $2/n = 1$.

8 FURTHER EXTENSIONS TO n-SPACE, AND APPLICATIONS

Areas of spherical zones and cross sections.

Part (a) of the next theorem extends the relation illustrated in Figure 18b, and part (b) extends that in Figure 2a. These extensions refer to cross-sections of an n-cylindroid cut by an $(n-1)$-dimensional hyperplane perpendicular to the axis of symmetry of the cylindroid. A zone of a sphere or a cylindroid is the part of its surface between two parallel cross-sections.

Theorem 4. (a) *The surface areas of the corresponding zones of an n-sphere and its circumscribing n-cylindroid are equal.*

(b) *Corresponding cross-sections of an n-sphere and its punctured circumscribing n-cylindroid have equal areas.*

Proof of (a). In Figure 21b, the zone on the hemisphere is in area equilibrium with the projected slice of a circular disk. When these are rotated through 4-space around the balancing axis, the hemispherical zone produces a zone of a 4-sphere, and the projected slice produces a corresponding zone of the circumscribing 4-cylindroid. By the balance-revolution principle, these two zones have equal areas. This proves (a) when $n = 4$, and the same type of argument works for general n.

Proof of (b). Archimedes' chord-by-chord balancing of a triangle and semicircular disk in Figure 1a can be extended to 3-space for a cone and hemisphere obtained by rotating the diagram in Figure 1a about a horizontal axis.

This balancing is indicated in Figure 27a which shows a cross-section of the cone and hemisphere through the axis of symmetry. Puncture both the cone and hemisphere with an arbitrary collection of chords, as depicted by the examples in Figure

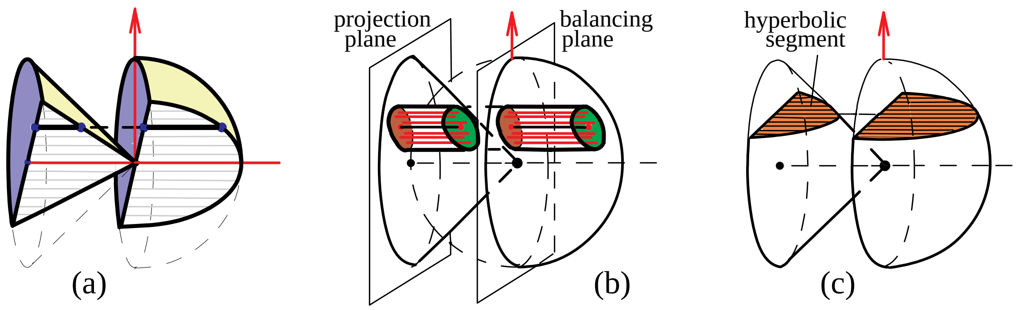

Figure 27: Extension of Archimedes' balancing in Figure 1a: (a) chords, (b) cables, (c) cross-sections of hemisphere and cone. The balance in (c) leads to equality of cross-sectional areas of a 4-sphere and its punctured circumscribing 4-cylindroid.

27b and 27c. Then the punctured parts will be in balance with respect to the central balancing plane. To see why, imagine two portions of a cable built from wires (the chords of the triangle and semicircle). Each pair of wires is in balance, therefore their union (the two portions of the cable) are also in balance. In particular, each horizontal cross-sectional area of the cone will be in area balance with the corresponding cross-sectional area of the hemisphere, as illustrated in Figure 27c.

Rotate Figure 27c through 4-space, around a balancing axis perpendicular to the axis of symmetry and perpendicular to the 4th coordinate axis. The rotated cone sweeps out a punctured 4-cylindroid, and the hemisphere sweeps out a 4-sphere. Each cross-section of the 3-cone will sweep out a cross-section of the punctured 4-cylindroid, and the cross-section of the 3-hemisphere will sweep out the corresponding cross-section of the 4-sphere. Because the two cross-sectional areas are in balance before revolution, after revolution the swept cross-sectional hyperareas are equal according to the balance-revolution principle. This proves (b) when n=4, and the same type of argument works for general n.

Recursion formulas for volume and surface area of n-spheres.

Let V_n and S_n denote the volume and surface area, respectively, of an n-sphere of radius r. The following formulas are consequences of Theorem 3a and the moment-volume principle:

$$V_{n+1} = \frac{2\pi r^2}{n+1} V_{n-1}, \tag{10}$$

$$S_{n+2} = \frac{2\pi r^2}{n} S_n = 2\pi r V_n, \tag{11}$$

$$\frac{n+1}{n} \cdot \frac{V_{n+1}}{V_n} = \frac{V_{n-1}}{V_{n-2}}, \tag{12}$$

$$\frac{S_{n+2}}{S_{n+1}} = \frac{V_n}{V_{n-1}}. \tag{13}$$

Initial values are $V_1 = 2r, V_2 = \pi r^2, S_1 = 2, S_2 = 2\pi r$.

Proof. By Theorem 3a, V_n is equal to the volume of the punctured n-cylindroid of radius r, which, by the moment-volume principle, is $2\pi M_{n-1}(\text{cone})$, where $M_{n-1}(\text{cone})$ is the volume moment of the $(n-1)$-cone of altitude r that sweeps out the punctured n-cylindroid. Therefore, using (9) with $h = r$ we find

$$V_n = 2\pi M_{n-1}(\text{cone}) = \frac{2\pi r^2}{n} V_{n-2}.$$

Now replace n by $n+1$ to get (10) which, in turn, implies (12). The circumsolid property $V_n = (r/n)S_n$ yields (11). From (11) we get $S_{n+2}/V_n = 2\pi r$, which is independent of n, hence $S_{n+2}/S_{n+1} = V_n/V_{n-1}$, which is (13).

Using the notation $v_n = V_n/2$ and $s_n = S_n/2$ we see that the recursions (10) through (13) also hold for n-hemispheres, whose centroids will be determined in the next section. Formulas (10) and (11) are known, but (12) and (13) are new.

Volume of n-dimensional cylindrical wedge.

The moment-wedge volume principle introduced in Section 4 for a cylindrical wedge can be extended to higher-dimensional space as follows:

Moment-wedge volume principle. *The "volume" W_{n+1} of a general $(n+1)$-cylindrical wedge and the moment M_n of its n-base are related by*

$$W_{n+1} = kM_n,$$

where k is a constant determined by the inclination of the truncating plane.

The moment-wedge volume principle has the following corollary:

The "volumes" of two $(n+1)$-cylindrical wedges with the same k have the same ratio as the moments of their bases. In particular, the "volumes" are equal if the moments are equal.

We apply the corollary to the balanced n-hemisphere and n-cone shown in the top row of Figure 25. Instead of rotating these objects we use them as bases of truncated $(n+1)$-cylinders with the same height h. Their "volumes" are equal and can be calculated explicitly as follows. The truncated cylinder whose base is the n-cone is an $(n+1)$-pyramid whose base can be regarded as a "rectangle" of height h whose "area" is $V_{n-1}h$, where V_{n-1} is the "volume" of the $(n-1)$-sphere of radius r. The "volume" of this pyramid is $W_{n+1} = V_{n-1}h \cdot r/(n+1) = V_{n+1}h/(2\pi r)$, according to recursion (10). Replacing $n+1$ by n we get the following theorem:

Theorem 5. *An n-cylindrical wedge of radius r and height h has volume W_n and lateral surface area A_n given by*

$$W_n = \frac{k}{2\pi}V_n, \quad A_n = \frac{k}{2\pi}S_n, \; n = 2, 3, \ldots,$$

where $k = h/r$ and S_n is the surface area of an n-sphere of radius r.

When $n = 2$, the wedge is a triangle of area $W_2 = hr/2$ and altitude $A_2 = h$.

9 FORMULAS FOR CENTROIDS

This section applies our balancing relations to obtain formulas for locating centroids of various objects of interest.

1. Centroid of regular circumgonal arc. A regular circumgon has an axis of symmetry through the center of its incircle, so the centroid also lies on this axis, as in Figure 28a. To find distance c of the centroid from the incenter, rotate the circumgon so its symmetry axis is perpendicular to the balancing axis as in Figure 28b. Balancing means that $cL = rH$, where L is the length of the circumgonal arc

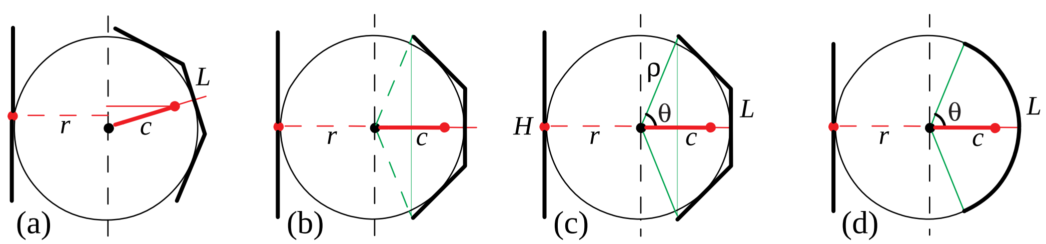

Figure 28: Rotating regular circumgon in (a) to obtain symmetric diagram in (b). (c) Relation between centroidal distance c and central angle subtended by the circumgon. (d) Limiting case when the circumgon becomes a circular arc.

and H is the length of its projection, hence

$$c = r\frac{H}{L}. \tag{14}$$

From Figure 28c we see that if the regular circumgon consists of n edges and is subtended by a central angle 2θ, with ρ the distance to a vertex as indicated, then $H = 2\rho \sin\theta$. A simple exercise shows that $L = 2\rho n \sin(\theta/n)$, and (14) gives us:

$$c = r\frac{\sin\theta}{n\sin\frac{\theta}{n}}. \tag{15}$$

Centroidal formula (14) for circumgonal arcs also holds for the limiting case when the circumgon is a circular arc of radius r with central angle 2θ, as in Figure 28d. In this case the projection has length $H = 2r\sin\theta$, the circular arc has length $L = 2r\theta$, so the centroidal distance of the arc from the axis, which we denote by $c(\theta)$, is obtained from (14) as

$$c(\theta) = r\frac{\sin\theta}{\theta}. \tag{16}$$

This also follows from (15) when $n \to \infty$. For a different derivation see [1; p. 11].

The next example uses (2) to determine area centroids.

2. Area centroid of circumgonal region with inradius r. When the projection region is a triangle as in Figure 6c, we have $P = Hr/2$, $c_P = 2r/3$, and (2) yields

$$c_A = \frac{r^2}{3}\frac{H}{A} = \frac{2}{3}r\frac{H}{L}, \tag{17}$$

because $A = rL/2$. When the circumgon with central angle 2θ is rotated so its axis of symmetry is horizontal as in Figure 28c, the foregoing becomes

$$c_A = \frac{2}{3}r\frac{\sin\theta}{n\sin\frac{\theta}{n}}. \tag{18}$$

In the limiting case when the circumgonal region is a circular sector of radius r and central angle 2θ, the area centroid lies on the bisector of the sector at a distance $C(\theta)$ from the center, where

$$C(\theta) = \frac{2}{3}r\frac{\sin\theta}{\theta}. \tag{19}$$

This follows from (17) using $H = 2r\sin\theta$, $L = r\theta$, or by letting $n \to \infty$ in (18). For a semicircular disk, $\theta = \pi/2$ and (19) gives $C(\pi/2) = 4r/(3\pi)$. This is the result Archimedes was seeking by balancing the triangle and semicircular disk in Figure 1b.

Next we use the double equilibrium in Figure 21a to treat the spherical wedge.

3. Area and volume centroids of spherical wedge. There are two centroidal distances associated with the wedge in Figure 21a: centroidal distance $C_A(\theta)$ for the spherical surface area A of the wedge, and $C_V(\theta)$ for the volume V of the wedge, where θ is the angle between the cutting plane and the equatorial plane. By symmetry, each of these centroids lies on the vertical plane bisecting the wedge.

Area centroid of spherical wedge. Area equilibrium means that

$$C_A(\theta) \cdot A = r \cdot E, \tag{20}$$

where E is the area of the ellipse. But we have $A = 4\pi r^2(\theta/\pi) = 4\theta r^2$ and $E = \pi r H/2 = \pi r^2 \sin\theta$. Using these in (20) and solving for $C_A(\theta)$ we obtain

$$C_A(\theta) = \frac{\pi}{4} r \frac{\sin\theta}{\theta}. \tag{21}$$

Volume centroid of spherical wedge. To find the distance of the volume centroid from the balancing plane we use the volume equilibrium relation:

$$C_V(\theta) \cdot V = \frac{3}{4} r \cdot V_E, \tag{22}$$

where V_E is the volume of the elliptical cone. In this case we have $V = 4\theta r^3/3$ and $V_E = Er/3 = \pi r^2 H/6 = (\pi/3) r^3 \sin\theta$, and when these are used in (22) we find

$$C_V(\theta) = \frac{3\pi}{16} r \frac{\sin\theta}{\theta}. \tag{23}$$

4. Volume centroid of spherical sector and of spherical segment. In Figure 20a, volume equilibrium about the balancing plane states that

$$C(\theta) V_s = \frac{3}{4} r V_c, \tag{24}$$

where $C(\theta)$ is the centroidal distance from the balancing plane, V_s is the volume of the spherical sector, and V_c is the volume of the cone. By (4) we have $V_s = 2\pi r^2 h/3$, where $h = r - r\cos\theta$ is the height of the spherical cap of the sector. The volume of the cone is $V_c = \pi(r\sin\theta)^2/3$. Using these in (24) and solving for $C(\theta)$ we find

$$C(\theta) = \frac{3}{8} r(1 + \cos\theta). \tag{25}$$

When the conical part is removed from the spherical sector in Figure 20a, the solid that remains is called a spherical segment. Archimedes in [5; Method, Prop. 9,

p. 35] determines the volume centroid of a spherical segment. His description for the distance c of the centroid from the center can be stated alternatively as follows:

$$c = \frac{3}{4}\frac{(r+h)^2}{2r+h}, \tag{26}$$

where $h = r\cos\theta$ is the altitude of the conical part of the sector of radius r.

The proof follows from the balancing relation involving volumes:

$$V_{\text{cone}} \cdot c_{\text{cone}} + V_{\text{segm}} \cdot c_{\text{segm}} = V_{\text{sect}} \cdot c_{\text{sect}}. \tag{27}$$

We know that $V_{\text{cone}} = \pi h(r^2 - h^2)/3$, $c_{\text{cone}} = 3h/4$, $V_{\text{sect}} = 2\pi r^2(r-h)/3$, and from (25), $c_{\text{sect}} = 3(r+h)/8$. Also, $V_{\text{segm}} = V_{\text{sect}} - V_{\text{cone}} = \pi(r-h)^2(2r+h)/3$. When these are used in (27) we obtain (26). In the special case when $h = 0$, (26) yields $c = 3r/8$ for the volume centroid of a hemisphere, which also follows from (23) with $\theta = \pi/2$.

5. Centroids of n-hemisphere. Let v_n, s_n denote, respectively, volume and surface area of an *n-hemisphere* of radius r, and let $c(v_n)$, $c(s_n)$ denote their respective centroidal distances from the center. Then the following remarkable recursions hold:

$$c(v_n) = \frac{n}{n+1}c(v_{n-2}), \quad c(s_{n+2}) = \frac{n}{n+1}c(s_n), \quad \text{and} \quad c(s_{n+2}) = c(v_n). \tag{28}$$

Proof. When an n-hemisphere of volume v_n and surface area s_n is rotated about a diameter, it generates an $(n+1)$-sphere of volume $2v_{n+1}$ and surface area $2s_{n+1}$. By the moment-volume principle (8) we have $2v_{n+1} = 2\pi v_n c(v_n)$ and $2s_{n+1} = 2\pi s_n c(s_n)$, from which we find

$$c(v_n) = \frac{v_{n+1}}{\pi v_n} \quad \text{and} \quad c(s_n) = \frac{s_{n+1}}{\pi s_n} = \frac{v_{n-1}}{\pi v_{n-2}},$$

the last relation coming from (11). Using (10) in these and invoking (12) we find

$$\frac{c(v_n)}{c(v_{n-2})} = \frac{c(s_{n+2})}{c(s_n)} = \frac{v_{n+1}}{v_{n-1}} \cdot \frac{v_{n-2}}{v_n} = \frac{n}{n+1}.$$

This proves the first two recursions in (28). The third follows from the first two by using the joint recursion in (13).

Repeated use of $c(s_{n+2}) = c(v_n)$ leads to the following explicit formulas for $n \geq 1$:

$$c(s_{n+2}) = c(v_n) = \frac{n!!}{(n+1)!!}p_n, \tag{29}$$

where $p_n = c(s_1) = r$ for odd n, $p_n = c(s_2) = 2r/\pi$ for even n, and $n!!$ is the double factorial symbol. Note that both $c(v_n)$ and $c(s_n)$ are rational multiples of r for odd n, and of r/π for even n.

It is easy to see that the following unexpected reciprocal recursions also hold:

$$c(v_{n+1}) = \frac{2r^2}{\pi(n+2)} \frac{1}{c(v_n)}, \quad c(s_{n+1}) = \frac{2r^2}{\pi n} \frac{1}{c(s_n)}, \tag{30}$$

with initial values $c(v_1) = r/2$ and $c(s_1) = r$. Unlike those in (28), they relate centroidal distances in consecutive dimensions.

To the best of our knowledge, centroidal formulas (28)-(30) for higher-dimensional hemispheres have not been previously published. Those in (28) are also valid for a general n-dimensional spherical wedge (the hemisphere being a special case), as well as for any n-dimensional cylindrical wedge. These results will be published elsewhere.

10 CONCLUDING REMARKS

Archimedes' relation on the volume and surface area of a sphere and its smallest circumscribing cylinder consists of two parts: (1) their volumes and surface areas have the same ratio, and (2) this ratio is 2/3. Theorem 2 in [4] tells us that part (1) is true not only for the sphere and cylinder but for any two circumsolids having the same insphere. *The ratio of their volumes is equal to the ratio of their outer surface areas.* This is true in n-space as well, with the ratio depending on the choice of circumsolids. For the n-sphere and n-cylindroid the ratio is $2/n$, which provides an elegant extension of Archimedes' result to n-space. But for the n-sphere and other circumsolids, such as the n-cylinder or n-cone, the ratio has no such simple form. The ratio $2/n$ emerges in Theorem 3 because the volume of an n-cylindroid bears a simple relation to the volume V_{n-2} of a sphere having 2 lower dimensions: $V_n(\text{cylindroid}) = \pi r^2 V_{n-2}$. Hence $V_n/V_n(\text{cylindroid}) = V_n/(\pi r^2 V_{n-2}) = 2/n$ by recursion (10). Other circumsolids, such as the n-cylinder and n-cone, have volumes simply related to the volume V_{n-1} of a sphere of one lower dimension, so comparing these volumes with that of an n-sphere is the same as comparing volumes of spheres of *consecutive* dimensions, for which there is no simple recursion.

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Abstract. Archimedes' mechanical balancing methods led to stunning discoveries concerning the volume of a sphere, and of a cylindrical wedge. This paper introduces new balancing principles, including double equilibrium, that go much further: they yield not only Archimedes' volume results but also a host of surprising relations involving both volumes and surface areas of circumsolids of revolution, as well as higher-dimensional spheres, cylindroids, spherical wedges, and cylindrical wedges. The concept of cylindroid, introduced here, is crucial for extending to higher dimensions Archimedes' classical relations on the sphere and cylinder. We also provide formulas for centroids of various portions of these objects, including remarkable new results for hemispheres in n-space. Throughout the paper we adhere to Archimedes' style of reducing properties of complicated objects to those of simpler objects.