
Complete Dissections: Converting Regions and Their Boundaries

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Abstract. Classical dissections convert any planar polygonal region onto any other polygonal region having the same area. If two convex polygonal regions are isoperimetric, that is, have equal areas and equal perimeters, our main result states that there is always a dissection, called a *complete dissection*, that converts not only the regions but also their boundaries onto one another. The proof is constructive and provides a general method for complete dissection using frames of constant width. This leads to a new object of study: isoperimetric polygonal frames, for which we show that a complete dissection of one convex polygonal frame onto any other always exists. We also show that every complete dissection can be done without flipping any of the pieces.

Dedicated to the memory of Martin Gardner

1. INTRODUCTION. DISSECTIONS INVOLVING BOUNDARIES. It is well known that any planar polygonal region can be dissected into smaller polygonal pieces that can be rearranged to form any other polygonal region of equal area (see for example [4, p. 221]).

What happens to the boundaries in these standard dissections?

Figure 1a shows a dissection of a triangle onto a rectangle, and Figure 1b shows a dissection of one rectangle onto another rectangle of prescribed altitude. In both examples, part of the boundary of one polygon ends up inside the other. This is to be expected, because the initial and final shapes have different perimeters. Figure 20a shows an even more dramatic example, a famous hinged dissection of Dudeney in which the entire boundary of a square ends up inside a triangle.

We are interested in dissections that not only preserve areas but also *convert the boundaries* onto each other as well. We call these *complete dissections*. They require that both regions have equal areas *and* equal perimeters. Such regions, called *isoperimetric*, were introduced in [1], and provide a natural motivation for this paper. To the best of our knowledge, such dissections have not been previously treated.



Figure 1. Dissection (a) of a triangle onto a rectangle; (b) of a rectangle onto another rectangle of prescribed altitude. Dark lines show how the boundaries are transformed.

In Figure 2, the triangle and rectangle are isoperimetric, but the dissection shown is not complete because one boundary is not converted entirely onto the other. In fact, there is no reason to expect that a complete dissection exists even though the figures are isoperimetric. Nevertheless, our first theorem reveals a surprising and profound result: for any two isoperimetric convex polygonal regions, a complete dissection always exists. Moreover, the proof shows how to construct such a dissection.

<http://dx.doi.org/10.4169/amer.math.monthly.118.09.789>

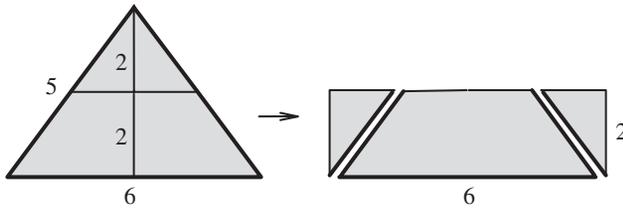


Figure 2. Standard dissection converting a triangle onto an isoparametric rectangle.

Before proceeding further, the reader might try to find a complete dissection that converts the triangle in Figure 3 onto the isoparametric isosceles trapezoid shown. This is a simpler task than for the pair in Figure 2.

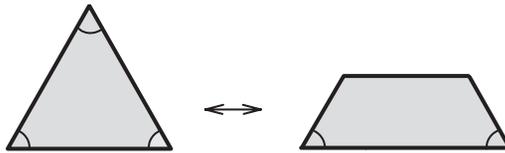


Figure 3. Equilateral triangle and an isoparametric trapezoid.

Figure 4 shows a complete dissection of any isosceles triangle onto an isosceles trapezoid, and of a general triangle onto a trapezoid with the same base angles.

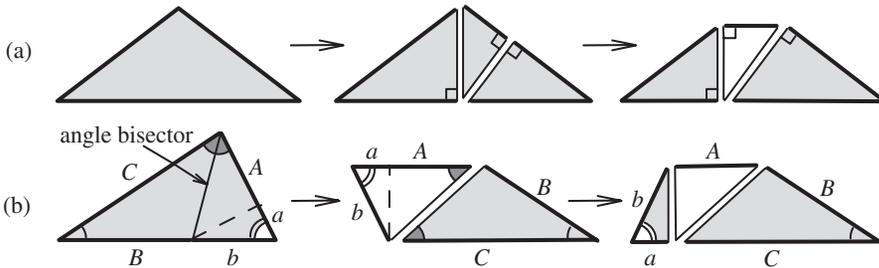


Figure 4. Complete dissection converting (a) isosceles triangle onto isosceles trapezoid, and (b) any triangle onto a trapezoid. In (a), the unshaded piece has been flipped. In (b) one piece is flipped and divided into two right triangles, the smaller of which is flipped again.

2. COMPLETE DISSECTION OF POLYGONAL REGIONS. We turn now to the first principal result of this paper.

Theorem 1. Any two isoparametric convex polygonal regions can be converted onto one another by complete dissection.

Proof. Consider two isoparametric convex polygonal regions A and B . Figure 5 shows an example with A triangular and B quadrilateral. This example displays all the essential features required in treating general convex isoparametric polygonal regions. The method of proof is suggested by an oversimplified intuitive idea: Remove each boundary and perform a standard dissection of the interior of A to produce the interior of B . Then restore the two boundaries to obtain the complete dissection.

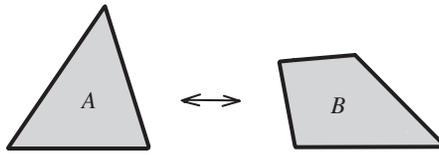


Figure 5. Isoparametric triangular and quadrilateral regions.

To make this intuitive idea rigorous, refer to Figure 6a, which shows a “frame” of some constant width w protruding into each region along its boundary. Choose w small enough so that the inner boundary of each frame will be a simple closed polygon with the same number of sides. Each region now consists of two parts: the frame plus the interior region surrounded by the frame. Keep in mind that:

The sum of areas, frame plus interior, is the same for both regions A and B.

Now unfold each frame at the outer vertices (thought of as hinges) and lay it out horizontally, as shown in Figure 6b. To be specific, use angle bisectors to cut each frame into trapezoidal pieces with isosceles triangular gaps between adjacent pieces.

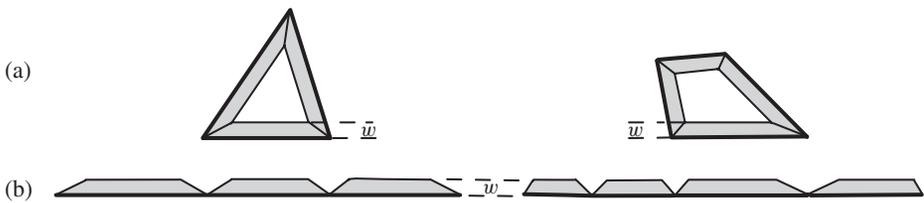


Figure 6. Frames unfolded to form adjacent trapezoidal pieces with triangular gaps.

Next, use standard dissections (as in Figure 1) of the regions interior to the frames in Figure 6a to convert them onto two rectangles with common altitude w (the frame width), as indicated in Figure 7a. Each rectangle has area equal to that of the interior that produced it, but, of course, the two rectangular areas are not equal to each other.

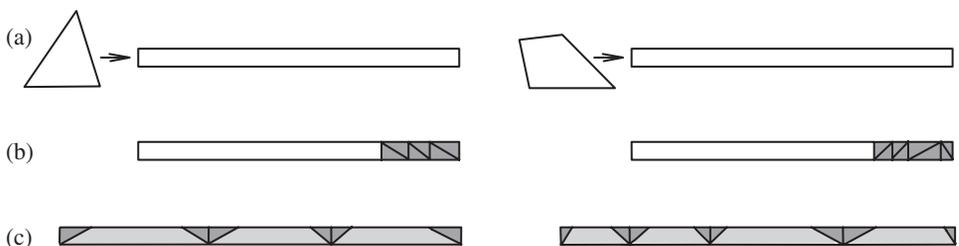


Figure 7. (a) Dissection of interior regions produces two rectangles of equal altitude, but not of equal area. Removing shaded triangular pieces in (b) to fill the gaps as shown in (c) leaves two unshaded congruent rectangles in (b).

From each of these rectangles, remove smaller triangles as needed (see Figure 7b) to fill the triangular gaps in Figure 6b. This transforms the unfolded frames into two congruent rectangles shown in Figure 7c, whose lower bases are the unfolded boundaries of A and B . By overlapping these rectangles we obtain their common dissection (dissection 1) which also converts the full boundary of A onto the full boundary of B .

The leftover unshaded rectangles in Figure 7b are also congruent because the sum of areas, frame plus interior, is the same for both regions in Figure 6a. By overlapping the unshaded rectangles in Figure 7b, we obtain a common dissection (dissection 2) of the two interior dissections inherited from Figure 7a. The union of common dissections 1 and 2 gives a complete dissection of region A onto region B and completes the proof. ■

Obviously, these dissections can be done in many different ways. For example, we can choose a different frame width w .

3. COMPLETE DISSECTION OF POLYGONAL FRAMES. The frames of constant width used in the proof of Theorem 1 lead us to consider the problem of complete dissection of such frames, where now both inner and outer boundaries are subject to conversion. Although frames are more complicated objects than those in Theorem 1, our second principal result (Theorem 2) states that any two isoparametric frames can also be converted onto one another by complete dissection. Before discussing Theorem 2, we explain precisely what we mean by a polygonal frame.

In this paper, the term *polygonal frame* refers to a frame of constant width. It has parallel inner and outer boundaries with constant distance separating the parallel edges. We restrict our discussion to *convex* polygonal frames, that is, frames in which both the inner and outer polygons are convex.

More precisely, start with a convex n -gon as inner boundary, and any frame width w . Draw lines outside the n -gon parallel to its sides at distance w from the sides. Segments of the lines will form another convex n -gon outside the inner boundary, like the examples in Figure 6a. The frame consists of the region between the two n -gons, including both boundaries. When unfolded at the outer vertices, the frame forms a set of adjacent trapezoidal regions akin to those in Figure 6b.

Complete dissections require isoparametric frames, which have equal areas and equal total perimeters (inner plus outer). Figure 8 shows a charming example (taken from [1]): a Pythagorean 3:4:5 triangular frame of constant width $w = 2$, and a square frame of the same constant width.

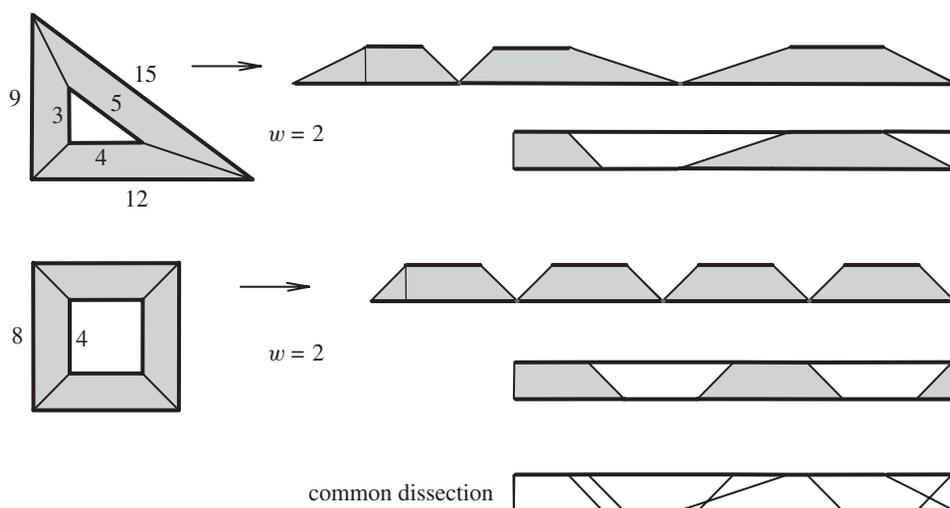


Figure 8. Complete dissections of two special isoparametric frames.

Note that both the areas and total perimeters of these frames have the same numerical value, 48. This is not merely a coincidence, but is a consequence of the following lemma when $w = 2$.

Lemma 1. *For any convex polygonal frame, its width w , area A , and total perimeter P are related by the equation*

$$A = \frac{1}{2}Pw. \quad (1)$$

The proof follows easily by applying the area formula for adjacent trapezoids forming the frame.

As an immediate consequence of Lemma 1 we have the following crucial result:

Corollary 1. *All isoparametric convex polygonal frames have the same width.*

Now return to Figure 8, which shows an unfolding of each frame into trapezoids of constant altitude $w = 2$, followed by a dissection onto a rectangle of the same altitude with two horizontal bases, the sum of whose lengths is the total perimeter of the frame. Because the two frames are isoparametric, so are the two rectangles. At the bottom of Figure 8, the dissected rectangles are superimposed to obtain a common dissection of the two frames that converts their total boundaries, indicated by the heavy lines.

The same type of argument can be applied to any pair of isoparametric convex polygonal frames to prove the following theorem.

Theorem 2. *Any two convex isoparametric polygonal frames can be converted onto one another by complete dissection.*

4. COMPLETE DISSECTIONS WITHOUT FLIPPING. In the foregoing dissections, some pieces may have been flipped. Although flipping might reduce the number of dissection pieces, in some applications, such as skin grafting or laying out carpeting, flipping is undesirable. It is known that every standard dissection can be carried out without flipping. Now we state our third principal result:

Theorem 3. *Every complete polygonal dissection can be done without flipping.*

Flipping a piece turns it into its mirror image, so the proof of Theorem 3 reduces to showing that a complete dissection of a general polygonal piece onto its mirror image can be done without flipping and in such a way that each edge of the polygon is converted to the corresponding edge of the mirror image. Such a dissection requires more than completeness, and we call it *strongly* complete. Now we will prove that every polygon can be converted onto its mirror image by using strongly complete dissection.

First dissect the polygonal piece into triangles, as illustrated in Figure 9. Then perform a strongly complete dissection on each triangle.

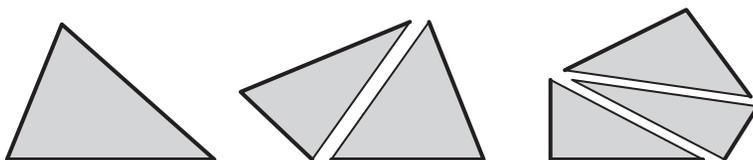


Figure 9. Any polygonal piece can be cut into triangles.

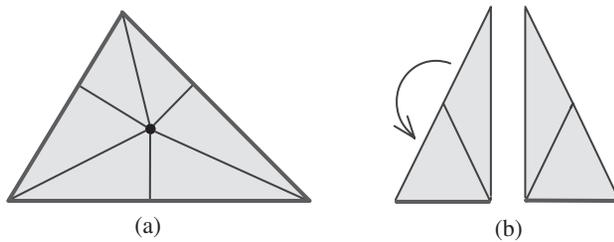


Figure 10. (a) Cutting a general triangle into right triangles, each of which has only one leg subject to conversion. (b) Converting one leg of a right triangle onto its mirror image.

To do this, we can dissect each triangle into six right triangles as shown in Figure 10a, where now each right triangle has only one leg as part of the boundary of the original triangle. This reduces the problem to that of dissecting a right triangle onto its mirror image, without flipping, so that one leg gets converted onto its mirror image. Figure 10b shows how a right triangle can be cut into two isosceles triangles. Rotate one of them to convert the given leg as shown. This completes the proof of Theorem 3.

By combining Theorems 1 and 3 we conclude that any two isoparametric convex polygons can be converted onto one another by complete dissection without flipping.

Examples. Figure 11 (taken from [1]) shows how easy it is to produce an endless supply of incongruent isoparametric pairs in which boundaries are transformed onto one another. In each case, a chord bisects the first region and one piece is flipped, thus giving a simple but complete dissection of one onto the other. It is reassuring to realize that every example of such a dissection can also be done without flipping.

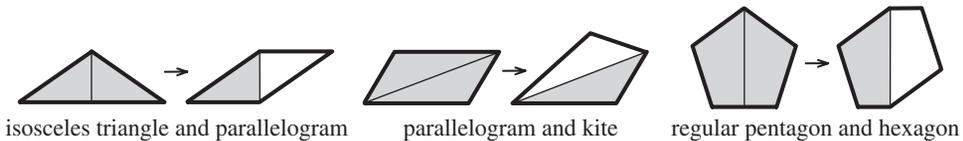


Figure 11. Isoparametric regions with boundaries converted onto one another.

Figure 12 shows two different complete dissections converting two isoparametric curvilinear regions onto one another. In the left figure (taken from [1]), a chord bisects the oval and one piece is flipped as in the examples of Figure 11 to produce the symmetric heart-shaped figure. A complete dissection without flipping is shown on the right, where cuts are made along the diagonals of the square inscribed in the oval, and then the pieces are rotated as shown. This dissection works for any oval having two perpendicular axes of symmetry.

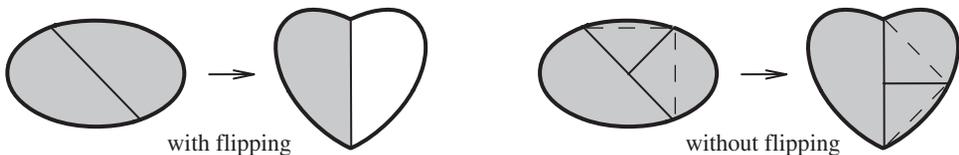


Figure 12. Two complete dissections of isoparametric curvilinear regions.

Figure 13 shows a complete dissection without flipping of a rectangular frame onto an isoparametric square frame. A similar dissection also works for any two rectangular frames of equal width having equal outer perimeters.

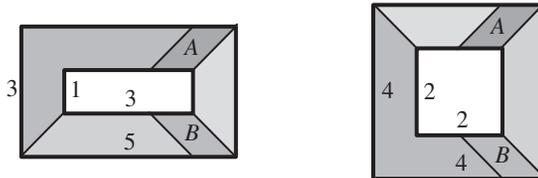


Figure 13. Complete dissection of rectangular frame and isoparametric square frame.

5. COMPLETE DISSECTIONS USED TO APPROXIMATE CURVILINEAR REGIONS. Figure 14 shows a frame that is partially polygonal and partially curvilinear. This example consists of a quadrilateral portion together with a curvilinear portion obtained as a limit of a portion of a polygonal frame of the same constant width w .

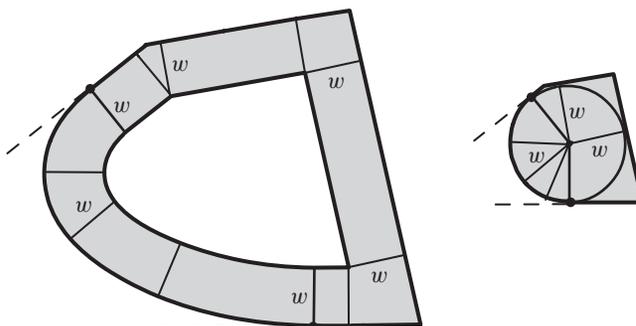


Figure 14. Frame with partially curvilinear boundaries, and a circumgon.

The second region in Figure 14 is a circumgon of inradius w that can be regarded as a degenerate case of the frame on the left, with the inner boundary replaced by a single point and the curved portion being a circular arc. Circumgons were introduced in [2], where it was shown that the area A and perimeter P of any circumgonal region with inradius w are related by the equation $A = Pw/2$, which is (1) in Lemma 1.

More general frames, partially curvilinear and partially polygonal, can be introduced similarly, so that they share formula (1) of Lemma 1. Then Corollary 1 implies that any two isoparametric partially curvilinear frames have the same width.

Figure 15 shows an example in which the curvilinear part is semicircular and the polygonal part is rectangular. This partially semicircular frame and the square frame shown adjacent to it are isoparametric. Actually, this square frame is only one of infinitely many rectangular frames of width $w = 2$ isoparametric to this semicircular

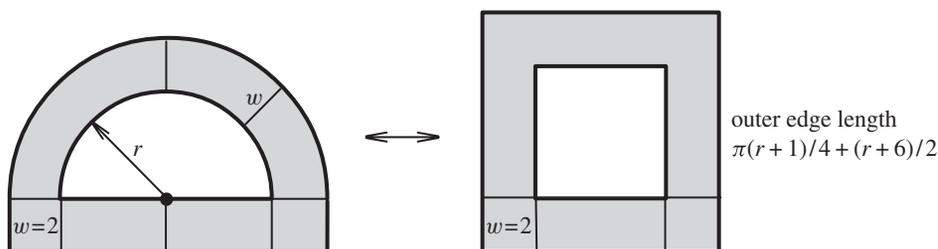


Figure 15. A partially semicircular frame and an isoparametric square frame.

frame. If the inner radius of the semicircular part is r , the common value of both the area and the perimeter is given by $A = P = 2\pi(r + 1) + 4(r + 2)$.

Further examples. Figure 16a (borrowed from [1]) shows two isoparametric circular sectors that are also isoparametric to the *same rectangle*. Figure 16b shows a circular frame and an isoparametric square frame of the same width.

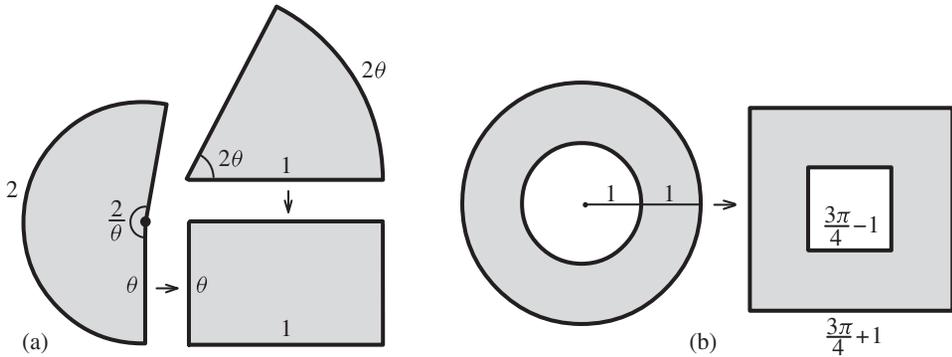


Figure 16. Partly circular regions isoparametric to polygonal regions.

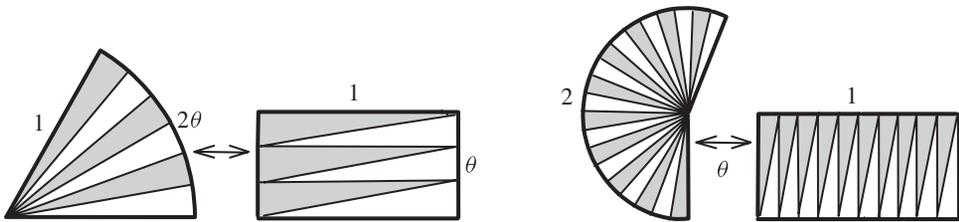


Figure 17. Radial slicing of two special circular sectors with their boundaries, rearranged differently to approximate the same isoparametric rectangle.

Figure 17 shows how the circular sectors in Figure 16a can be dissected into an even number of radial slices that can be rearranged to form a figure approximating the same rectangle. In one case the slices are arranged in horizontal layers, and in the other case in vertical layers. As the number of slices increases without bound, both approximations have the same rectangle as a limit, with the curved boundaries becoming the rectangular boundaries, in the style of Archimedes. Figure 18 shows a dissection of the circular frame in Figure 16b that approximates the square frame. The

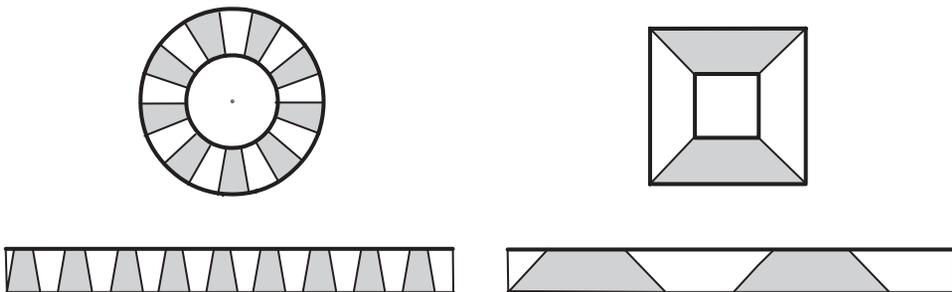


Figure 18. Radial slices of a circular frame and its boundary, rearranged to approximate the isoparametric square frame.

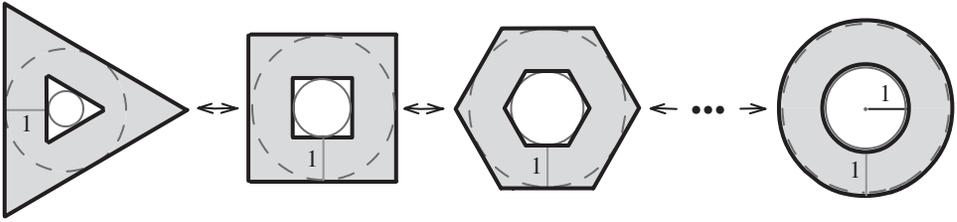


Figure 19. Regular polygonal frames and a circular frame, all isoparametric.

two frames in Figure 16b are members of an infinite family of isoparametric regular frames of constant width $w = 1$, examples of which are shown in Figure 19.

Each inner regular n -gon in this family has an incircle of diameter d_n , where

$$d_n = \frac{3\pi}{n \tan \frac{\pi}{n}} - 1,$$

and each outer regular n -gon has an incircle of diameter $D_n = d_n + 2w = d_n + 2$.

6. CONCLUDING REMARKS. This paper introduces a new development in classical dissection problems: complete dissections that convert regions and their boundaries onto one another. This requires isoparametric regions, objects of independent interest introduced in [1].

The methods of this paper can be adapted to more general situations, illustrated by the example in Figure 20b, a square and equilateral triangle with equal areas and unequal perimeters. Using a method similar to that for proving Theorem 1, with the frames in Figure 6 replaced by frames like those in Figure 20c, it is possible to dissect a general polygonal region onto any other polygonal region of equal area and unequal perimeter, with the shorter boundary converted onto part of the longer boundary, as indicated by the heavy lines. This suggests further modifications of classical dissections that might provide interesting applications to recreational mathematics.

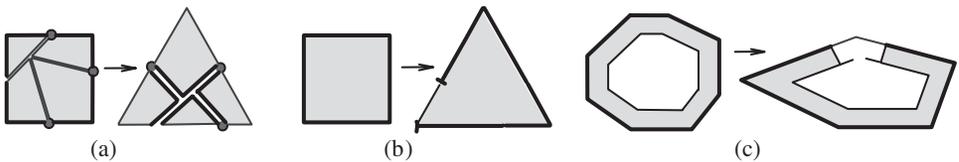


Figure 20. In (a) and (b), the areas are equal and the perimeters are unequal. (c) Modifying the frames in Figure 6 to treat dissections of the type in (b).

In view of Max Dehn's counterexample that settled Hilbert's third problem, extensions of our ideas to dissections of solids in 3-space are not always possible. Nevertheless, circumsolid shells, introduced in [3], have properties analogous to polygonal frames. For example, all circumsolid shells have constant thickness (see Theorem 8a in [3]). The analog of Lemma 1 is given by Theorem 10 in [3].

Frederickson's book [4] gives an admirable introduction to the field of geometric dissections, and contains a valuable bibliography of known results. Although there is a vast literature on standard polygonal dissection, we were not able to find any references relating to complete dissections of the type discussed in this paper.

It is surprising that complete dissections have not been previously discussed in connection with cake slicing. Here's a natural question: Can we cut a cake with white icing on top and chocolate icing on its outer edges and rearrange the pieces to form a cake of another shape (as in Figure 12) so that the white icing stays on top and the chocolate icing stays on the outer edges of the rearranged cake? Theorem 3 shows that this can always be done for polygonal cakes.

ACKNOWLEDGMENTS. The authors wish to thank the referees for valuable suggestions that improved this paper. They also wish to thank Dan Velleman for suggesting a way to considerably shorten our proof of Theorem 3, and for suggesting the simple example of isoparametric frames in Figure 13.

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