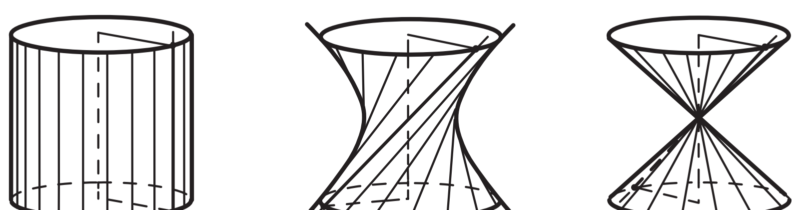
New Descriptions of Conics via Twisted Cylinders, Focal Disks, and Directors

Tom M. Apostol and Mamikon A. Mnatsakanian

1. INTRODUCTION. Conics have been investigated since ancient times as sections of a circular cone. Surprising descriptions of these curves are revealed by investigating them as sections of a hyperboloid of revolution, referred to here as a *twisted cylinder*. We generalize the classical focus-directrix property of conics by what we call the *focal disk-director property* (Section 2). We also generalize the classical bifocal properties of central conics by the *bifocal disk property* (Section 5), which applies to all conics, including the parabola. Our main result (Theorem 5) is that each of these two generalized properties is satisfied by sections of a twisted cylinder, and by no other curves. Although some of these results are mentioned in Salmon's treatise [**6**] and in a note by Ferguson [**4**], they are not widely known, and we go far beyond these earlier treatments.

Twisted cylinders. A circular cylinder is a ruled surface with its generators parallel to the axis of the cylinder. Figure 1(a) shows a portion of a cylinder between two circular bases. Rotate the lower circle about the axis to form a new ruled surface (Figure 1(b)), all of whose generators make the same angle with the axis. Because the constancy of this angle is fundamental in our analysis, we prefer to call the surface a twisted cylinder rather than a hyperboloid of revolution. The circular cylinder and cone are special cases. (See website [**5**] for interactive Java animation.) In this paper, all twisted cylinders have vertical axes.



(a) (b) (c)

Figure 1. Cylinder (a) and cone (c) as special cases of twisted cylinder (b).

In general, a section of a twisted cylinder by an inclined plane is analogous to a section of a cone (Figure 2). For a small angle of inclination, the section is an ellipse. When the cutting plane is tilted more to become parallel to a generator, the section is a parabola. Tilting it further produces a hyperbola.

Differences between sections of a cone and of a twisted cylinder. Significant differences are revealed when the cutting plane is translated. On a cone, translation of the cutting plane always produces a similar conic, with the same eccentricity. But on a twisted cylinder, as Figure 2(b) shows, a parabola b.1 can degenerate into a pair of parallel lines b.2 (which we call a *degenerate parabola*). More dramatic changes occur when the intersection is a hyperbola, c.1. Figure 2(c) shows two critical positions, c.2,

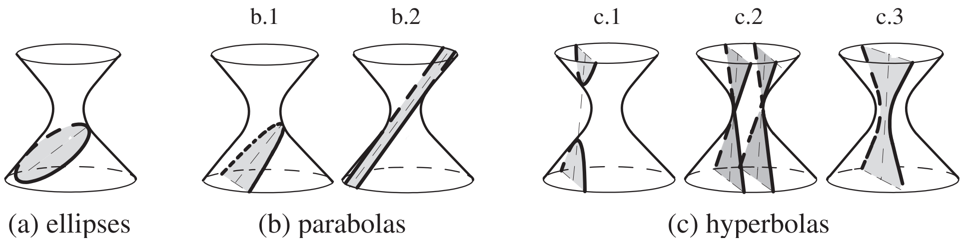


Figure 2. Intersections of a twisted cylinder with an inclined plane.

at which the plane is tangent to the twisted cylinder, and the hyperbola degenerates into a pair of intersecting lines. Between these critical positions there are intermediate flipped hyperbolas, c.3, whose eccentricity is *not* the same as that of the hyperbolas in c.1. Further translation flips the hyperbola again, to sections similar to those in c.1. On a cone, the critical positions coincide, and the flipped hyperbolas c.3 do not appear. In this paper, the term *conic* refers to any section of a twisted cylinder, including degenerate cases.

Focal disks. Inscribe a sphere inside a twisted cylinder so it intersects the cutting plane along a circular disk, shown shaded in Figure 3. This disk is in the plane of the conic, and we call it a *focal disk* for that conic.

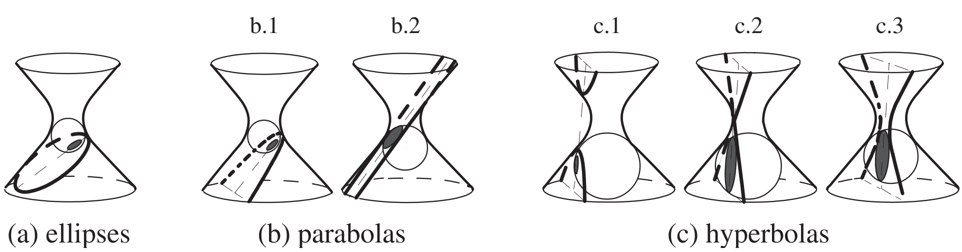


Figure 3. Inscribed sphere pierces the plane of the conic along a focal disk.

If the sphere happens to be tangent to the cutting plane, the focal disk is a point (which turns out to be a focus!). This can occur in Figure 3 in (a), b.1, and c.1 but not in b.2 and c.3. In c.3 the foci of the hyperbola are *outside* the twisted cylinder.

Families of focal disks. By moving the sphere upward or downward through the cutting plane, keeping it inscribed in the twisted cylinder, we obtain an infinite family of focal disks for a given conic. Examples in the plane of the conic are shown for an ellipse in Figure 4(a), a parabola in Figure 4(b), and a degenerate parabola in Figure 4(c). When the ellipse is a circle the focal disks are concentric with this circle.

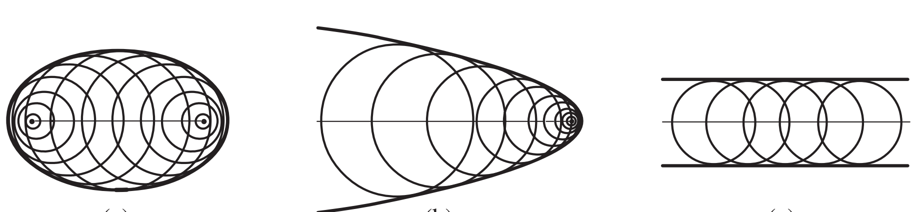


Figure 4. Families of focal disks associated with the conics in Figures 3(a) and 3(b).

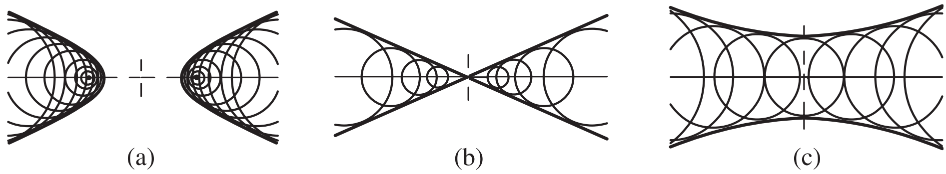


Figure 5. Families of focal disks associated with the sections in Figure 3(c). The family in (c) does not contain the foci of the hyperbola and cannot occur on a cone.

Figures 5(a), 5(b), and 5(c) show the focal disks obtained by moving an inscribed sphere in Figure 3(c), corresponding to the sections in c.1, c.2, and c.3.

Abnormal configurations. A conic with its focal disks defines a configuration. Those configurations in Figure 4(c) and Figure 5(c) we call *abnormal* to emphasize that they cannot occur on a cone. In these configurations a focal disk cannot be shrunk to a focus, and any circular disk tangent to both branches is a focal disk. The degenerate configuration in Figure 4(c) can be regarded as a limiting configuration of Figure 5(c). The configurations that can occur on a cone we call *normal*.

The rest of the paper is divided into three parts as follows:

Part 1: Focal disk-director description of noncircular conics (Sections 2–4).

Part 2: Bifocal disk description of conics (Sections 5–7).

Part 3: Supplementary results (Sections 8–10).

PART 1: FOCAL DISK-DIRECTOR DESCRIPTION OF NONCIRCULAR CONICS

2. DISK-DIRECTOR RATIO. FOCAL DISK-DIRECTOR PROPERTY. In Figure 6(a) the inscribed sphere touches the twisted cylinder along a circle we call a *terminator*. If the cutting plane is parallel to the plane of the terminator, the section is a circle and the focal disk is concentric with this circle. Otherwise, the planes intersect along a line we call a *director*. The section shown in Figure 6 is an ellipse, but it could be any noncircular conic.

In Figure 6(b), let P denote a typical point on the conic, let PT be the length of a tangent segment from P to the focal disk, and let PD be the distance from P to the director.

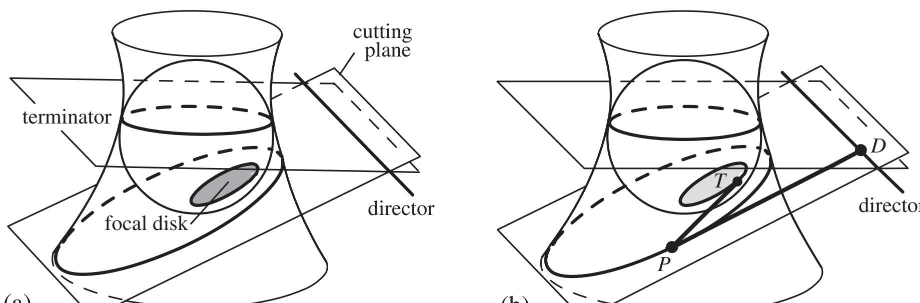


Figure 6. (a) Focal disk and director. (b) Disk-director ratio $q = PT/PD$ is constant.

We introduce the *disk-director ratio* $q = PT/PD$ and show that the conic has the following *focal disk-director property*:

Proposition 1. *On any noncircular conic of intersection, the disk-director ratio $q = PT/PD$ is constant, that is, independent of P.*

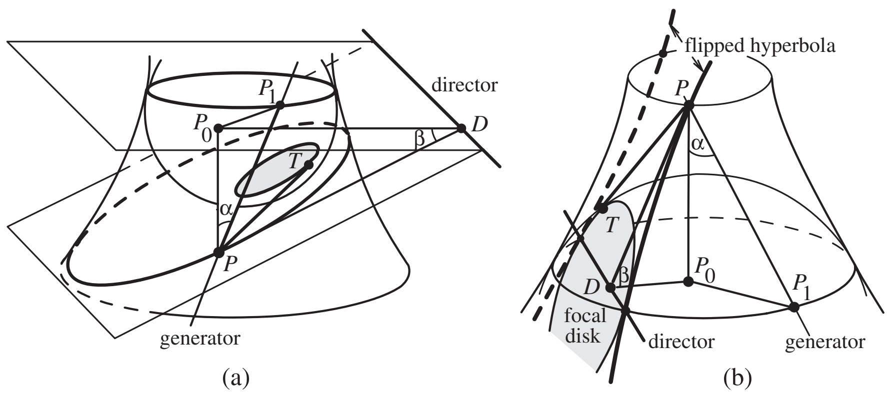


Figure 7. Proof of focal disk-director property: (a) ellipse; (b) flipped hyperbola.

Proof. In Figure 7, α denotes the angle formed by a generator and a line parallel to the axis of the twisted cylinder, and $\beta > 0$ denotes the angle between the cutting plane and a plane through the terminator. Let P_1 denote the point where the generator through P intersects the terminator, and let P_0 denote the vertical projection of P on the plane of the terminator. Then $PP_0/PP_1 = \cos\alpha$, and $PP_0/PD = \sin\beta$. But $PT = PP_1$ because both are lengths of tangent segments to the same sphere from an external point P. Therefore

$$q = \frac{PT}{PD} = \frac{PP_1}{PD} = \frac{PP_1}{PP_0}\frac{PP_0}{PD} = \frac{\sin\beta}{\cos\alpha}, \tag{1}$$

which is a constant independent of P. The case of a flipped hyperbola is shown in Figure 7(b). ■

Note. If $\beta = 0$ the cross section is a circle, the foregoing definition of director D as a line of intersection does not apply, and the disk-director ratio PT/PD is not defined. That is why Proposition 1 and subsequent results involving the disk-director ratio are restricted to noncircular conics.

Invariant properties of *q*. From (1) we see that the disk-director ratio q depends only on β, the angle of inclination of the cutting plane, and on α, the angle between a generator of the twisted cylinder and its vertical axis. Therefore q is invariant under any change in configuration that preserves these angles, such as moving the inscribed sphere upward or downward, translating the cutting plane, or scaling by similarity. When the twisted cylinder is a cone with vertex angle 2α, (1) is a known formula for the eccentricity of the conic section.

By moving the inscribed sphere through the cutting plane to produce families of focal disks, we obtain:

Proposition 2. *A noncircular conic of intersection has the focal disk-director property with respect to infinitely many focal disk-director pairs, all of which have the same disk-director ratio.*

3. DISK-DIRECTOR RATIO RELATED TO ECCENTRICITY. The classical focus-directrix property (Figure 8(b)) does not apply to circular conic sections, and likewise the generalized focal disk-director property does not apply to circular sections of a twisted cylinder. For a nonhorizontal cutting plane, the focal disk-director property in this plane is illustrated in Figure 8(a) for normal configurations. It states that there is a constant q such that $PT = q\,PD$ for every P on the conic. In normal cases, when the focal disk can shrink to a point (denoted by F in Figure 8(b)), the director becomes the classical directrix of a conic with focus F. The ratio $q = PT/PD$, which is unchanged in the shrinking process, turns into the classical eccentricity $e = PF/PD$ for a noncircular conic section. Therefore:

For normal noncircular configurations, the disk-director ratio is equal to the eccentricity of the conic.

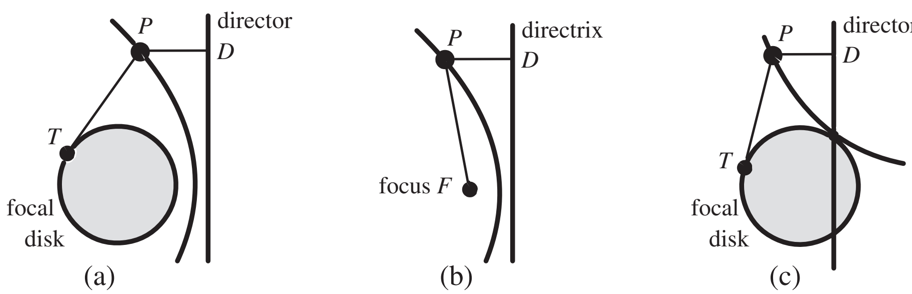


Figure 8. Focal disk-director property: $PT = q\,PD$ (a) normal configuration; (c) abnormal configuration. (b) Classical focus-directrix property: $PF = e\,PD$.

In an abnormal case shown in Figure 5(c), the disk cannot be shrunk to a focus (Figure 8(c)), and the foregoing argument does not apply. The transition from normal to abnormal occurs when the hyperbola c.1 in Figure 3(c) degenerates and then flips to the hyperbola c.3 which, as we will see in a moment, has similar asymptotes but not necessarily the same eccentricity. During this transition, q remains invariant and is equal to the eccentricity e of the hyperbola in c.1.

To determine the relation between q and the eccentricity ε of the flipped hyperbola with the same asymptotes it suffices to relate e and ε. We do this with the help of Figure 9.

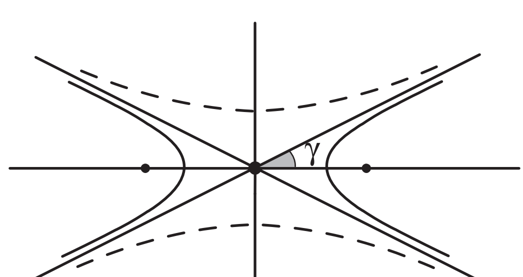


Figure 9. Hyperbola with horizontal focal axis has eccentricity $e = 1/\cos\gamma$. Flipped hyperbola with the same asymptotes has conjugate eccentricity $\varepsilon = 1/\sin\gamma$.

Conjugate eccentricities of flipped hyperbolas with the same asymptotes. In Figure 9, γ denotes the angle between an asymptote of a hyperbola of eccentricity e and its horizontal focal axis. It is easy to show that $e = 1/\cos\gamma$. (See, for example, Figure 13.14 in [**1**].) Therefore $q = 1/\cos\gamma$, so the angle between asymptotes is also invariant under translation of the cutting plane. But the eccentricity ε of the flipped hyperbola (dashed in Figure 9) with the same asymptotes is given by $\varepsilon = 1/\cos(\pi/2 - \gamma) = 1/\sin\gamma$. Hence

$$\frac{1}{e^2} + \frac{1}{\varepsilon^2} = 1, \quad \text{or} \quad e = \frac{\varepsilon}{\sqrt{\varepsilon^2 - 1}}.$$

We call e and ε *conjugate eccentricities*. They are equal only for a rectangular hyperbola, in which case the asymptotes are perpendicular and $e = \varepsilon = \sqrt{2}$.

Relation between the disk-director ratio and eccentricity. We know that $q = e$ for the configurations in Figures 5(a) and 5(c), where e is the eccentricity of the hyperbola in Figure 5(a). Consequently, for the flipped hyperbola in Figure 5(c), q is related to its eccentricity ε by the relation

$$q = \frac{\varepsilon}{\sqrt{\varepsilon^2 - 1}}.$$

Thus, we have established:

Proposition 3. *For normal noncircular configurations, the disk-director ratio is the eccentricity of the conic, and for abnormal configurations it is the conjugate eccentricity.*

4. FOCAL DISK-DIRECTOR THEOREM AND ITS CONVERSE.

Because *every* conic, including a flipped hyperbola, can be obtained as a section of some twisted cylinder, the foregoing results can be summarized as follows:

Theorem 1. *For any given noncircular conic, with given eccentricity, there is an infinite family of focal disk-director pairs, all with disk-director ratio equal to that eccentricity. For any hyperbola there is a second infinite family of focal disk-director pairs, all with disk-director ratio equal to its conjugate eccentricity.*

The following converse to Theorem 1 shows that *noncircular conics are the only curves having the focal disk-director property.*

Theorem 2. (a) *Suppose we are given a disk, a coplanar line L, and a positive number q. The locus of all points P in that plane such that the length of the tangent segment from P to the disk is q times the distance from P to L is a noncircular conic. The disk is a focal disk with director L, and q is the disk-director ratio.*

(b) *The conic in* (a) *has eccentricity $e = q$, except when director L intersects the focal disk and $q > r/\sqrt{r^2 - \lambda^2}$, where r is the radius of the disk, and $|\lambda| < r$ is the distance from the disk's center to L, in which case $e = q/\sqrt{q^2 - 1}$.*

Proof of (a). In Figure 10, choose the origin of xy coordinates at the center of the given disk of radius r, and take L parallel to the y axis with equation $x = \lambda$, where

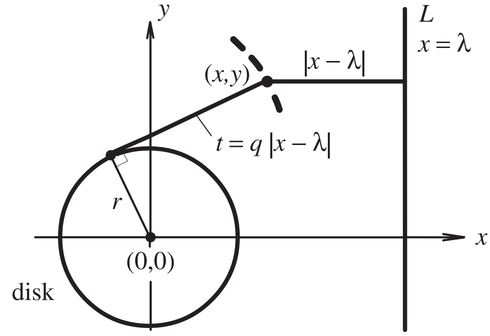


Figure 10. Diagram for the proof of Theorem 2(a).

$|\lambda| \geq 0$ is the distance from the disk's center to L. The length t of the tangent segment to the disk from point (x, y) on the locus satisfies the defining property

$$t = q\,|x - \lambda|, \tag{2}$$

which, together with $x^2 + y^2 = t^2 + r^2$, gives $x^2 + y^2 - r^2 = q^2(x^2 - 2\lambda x + \lambda^2)$ or

$$(1 - q^2)x^2 + y^2 + 2\lambda q^2 x = r^2 + q^2\lambda^2. \tag{3}$$

This represents a conic. The disk and line L constitute a focal disk-director pair with disk-director ratio q.

Proof of (b). $\boldsymbol{q = 1}$. In this case, (3) represents a parabola (or degenerate case) given by

$$y^2 = -2\lambda x + r^2 + \lambda^2. \tag{4}$$

When $\lambda = 0$, the parabola degenerates to the pair of horizontal lines $y^2 = r^2$, tangent to the focal disk as in Figure 4(c), with the director passing through the center of the disk. If $\lambda \neq 0$, (4) represents a nondegenerate parabola with $e = q = 1$.

$\boldsymbol{q \neq 1}$**.** By completing the squares in (3) we find the equation

$$(x - \rho)^2 + \frac{y^2}{1 - q^2} = \frac{B}{1 - q^2}, \tag{5}$$

which represents a *central conic*, with

$$\rho = -\frac{\lambda q^2}{1 - q^2}, \quad \text{and} \quad B = r^2 - \lambda\rho. \tag{6}$$

When $B = 0$, (5) becomes $y^2 = (q^2 - 1)(x - \rho)^2$, which implies $q > 1$ and represents a degenerate case of a hyperbola consisting of two lines intersecting at $(\rho, 0)$ with slopes $\pm\sqrt{q^2 - 1}$.

Now assume $B \neq 0$, so the central conic is a nondegenerate ellipse or hyperbola. From (5) we see that its center is at $(\rho, 0)$ which, by (6), is independent of r. To determine the eccentricity, first we find lengths a and b of the semiaxes as follows.

$\boldsymbol{0 < q < 1}$**.** In this case $B > 0$ and (5) can be written as $(x - \rho)^2/a^2 + y^2/b^2 = 1$, which represents a *noncircular ellipse* with

$$a^2 = \frac{B}{1-q^2}, \quad \text{and} \quad b^2 = (1-q^2)a^2. \tag{7}$$

This gives $q = \sqrt{1 - b^2/a^2}$, which is also the eccentricity, hence $e = q$.

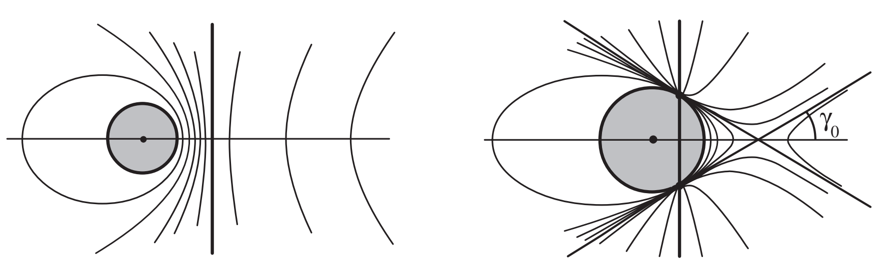
***q* > 1.** In this case (5) represents a *hyperbola* given by

$$(x-\rho)^2 - \frac{y^2}{q^2-1} = \frac{-B}{q^2-1}. \tag{8}$$

There are two types of hyperbolas, depending on whether $B < 0$ (horizontal focal axis) or $B > 0$ (vertical focal axis).

***B* < 0.** In this case $-B = b^2$ for some $b > 0$ and (8) can be written in the form $(x-\rho)^2/a^2 - y^2/b^2 = 1$, where $a^2 = b^2/(q^2-1)$. This hyperbola has its foci on the x axis and eccentricity $e = \sqrt{1 + b^2/a^2} = q$.

***B* > 0.** In this case we write $B = b^2$ for some $b > 0$ and (8) takes the form $y^2/b^2 - (x-\rho)^2/a^2 = 1$, where $a^2 = b^2/(q^2-1)$. The foci of this hyperbola are not on the $x$axis but on the vertical line $x = \rho$. The director $x = \lambda$ is also vertical. This is the flipped hyperbola in the abnormal case. The eccentricity of the flipped hyperbola is $e = \sqrt{1 + a^2/b^2} = q/\sqrt{q^2-1}$. Note that this cannot occur when $r = 0$, which is the classical case, so $B > 0$ requires $r > 0$. Examples for various q are shown in Figure 11. It is easy to verify that the case with $B > 0$ occurs when $\lambda^2 < r^2$ and $q > r/\sqrt{r^2 - \lambda^2} = 1/\cos\gamma_0$, where γ_0 is the asymptote angle in Figure 11(b). ∎



(a) (b)

Figure 11. Examples of the conic for various values of q. The director intersects the disk in (b) but not in (a).

Earlier we mentioned that the degenerate configuration in Figure 4(c) can be regarded as a limiting case of the abnormal configuration in Figure 5(c). In terms of eccentricity, this may seem paradoxical because Figure 4(c) is a limit of parabolic configurations, with all parabolas having eccentricity $e = 1$, but as the configuration in Figure 5(c) approaches that in Figure 4(c), the eccentricities of the hyperbolas tend to ∞. However, there is no paradox in terms of the disk-director ratio q because during this limit process $q \to 1$, which matches the value of q in Figure 4(c).

PART 2: BIFOCAL DISK DESCRIPTION OF CONICS

5. BIFOCAL DISK PROPERTY. Return to the twisted cylinder cut by an inclined plane (Figure 2), and inscribe in it *two* spheres as in Figure 12, each of which pierces a focal disk in the plane of the conic. We will prove the following *bifocal disk property*.

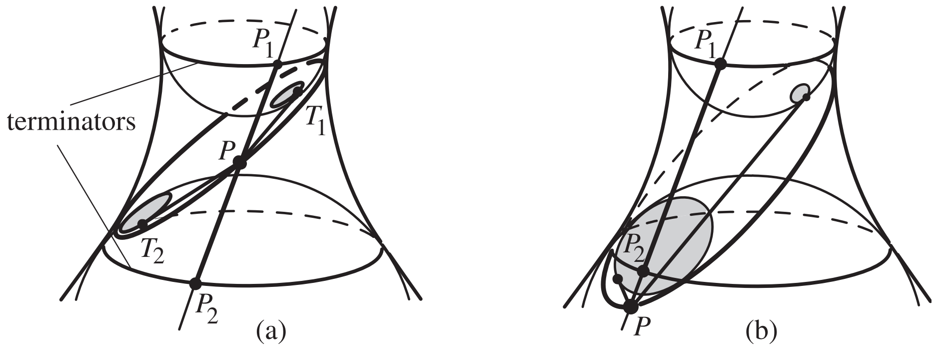


Figure 12. Bifocal disk property. In (a), $PT_1 + PT_2 = c$. In (b), $PT_1 - PT_2 = c$.

Proposition 4. *There is a constant c such that for each point P on the conic of intersection, either the sum or absolute difference of the tangent lengths from P to the two focal disks is equal to c.*

Proof. In Figure 12, ellipses are shown, but the argument applies to all conics, including circular cross sections. A generator of the twisted cylinder passing through a typical point P on the conic intersects the terminator of sphere 1 at P_1 and that of sphere 2 at P_2. The tangent segment PT_1 from P to disk 1 has the same length as the tangent segment PP_1 because both are tangent to sphere 1. Similarly, $PP_2 = PT_2$ for sphere 2. Therefore, if P is between the upper and lower terminators as in Figure 12(a), we have the sum

$$PT_2 + PT_1 = PP_2 + PP_1 = P_2P_1 = c,$$

where $c = P_2P_1$ is the constant distance between the terminators measured along the generator on the twisted cylinder. But if P lies below the lower terminator so that $PT_2 < PT_1$ as in Figure 12(b), the same argument shows that the difference

$$PT_1 - PT_2 = PP_1 - PP_2 = P_2P_1 = c,$$

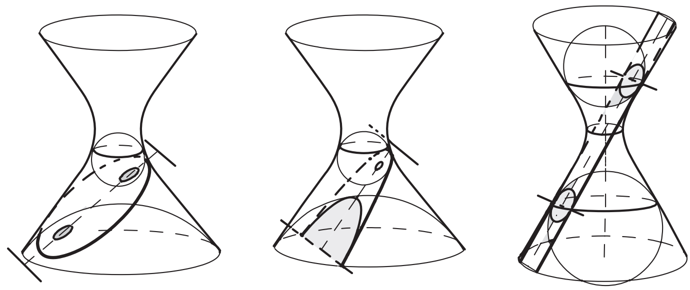
where c is the same constant distance above. Therefore, the sum or absolute difference of the tangent lengths (larger minus smaller) from a point on the conic of intersection to the two focal disks is constant. ■

Possible configurations. Figures 13 and 14 are elaborations of Figure 3, and show examples of some of the different configurations of conics and two focal disks (shown shaded) that can occur. The directors are also shown as lines of intersection of the cutting plane and the two terminator planes.

In Figure 13(c) the cutting plane has been translated to produce a degenerate parabola (two parallel lines). In [**2**] we show that this degenerate parabola leads to a surprising property of hyperbolas! Figure 14 relates focal disks to the hyperbolic cross sections shown in Figure 3(c). An abnormal case is depicted in Figure 14(c). Unlike the normal cases in Figures 13(a), 13(b), and 14(a), the disks in Figure 14(c) cannot be shrunk to become foci.

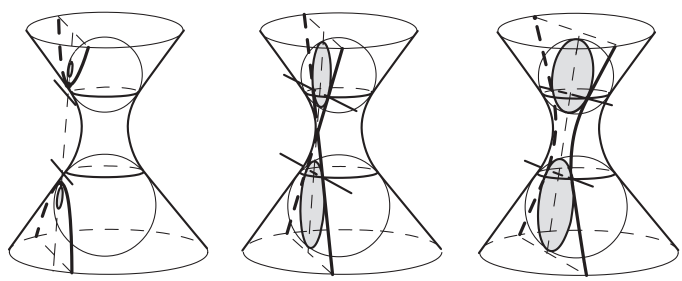
By moving the inscribed spheres independently, we obtain:

Proposition 5. *Every conic of intersection has the bifocal disk property with respect to infinitely many pairs of focal disks.*



(a) (b) (c)

Figure 13. (a) Ellipse. (b) Parabola. (c) Degenerate parabola obtained by translating the cutting plane in (b). This degenerate conic cannot be obtained as a section of a cone.



(a) (b) (c)

Figure 14. (a) Hyperbola. Translating the cutting plane in (a) produces a degenerate hyperbola (b). Further translation leads to the abnormal configuration in (c).

Geometric relations on a twisted cylinder. Figure 15 relates geometric parameters of the inscribed spheres, terminators, focal disks, and directors with the angles α and β introduced in Figure 7. Figure 15(a) shows that

$$c = s \cos \alpha, \tag{9}$$

where s is the distance between the centers of the inscribed spheres, and c is the length of the portion of the generator joining the terminators. Figure 15(b) gives

$$h = c \cos \alpha, \tag{10}$$

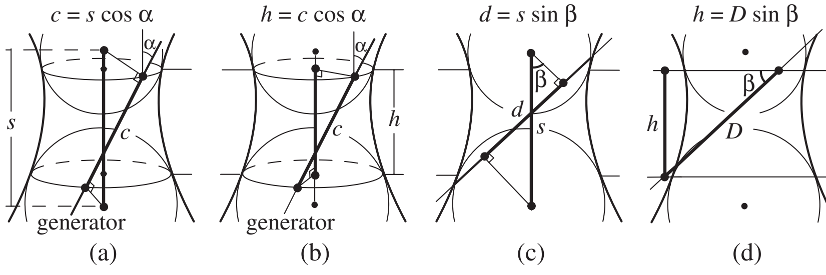


Figure 15. Relating parameters c, s, h, d, D with angles α and β.

where h is the distance between the planes of the terminators. In Figure 15(c),

$$d = s \sin \beta, \tag{11}$$

where d is the distance between centers of the focal disks. Note that $d = 0$ when $\beta = 0$, which means the focal disks are concentric when the cutting plane is horizontal. For $\beta > 0$ Figure 15(d) yields

$$h = D \sin \beta, \tag{12}$$

where D is the distance between the parallel directors of the focal disks.

Dividing (11) by (9) and using (1) we find

$$\frac{d}{c} = \frac{\sin \beta}{\cos \alpha} = q. \tag{13}$$

Because q is related to the eccentricity of a noncircular conic as described in Proposition 3, (13) shows that the ratio d/c bears the same relation to eccentricity in the focal disk-director property.

Proposition 6. *If $d > 0$, the ratio d/c of the distance d between centers of the focal disks and the constant sum or absolute difference c of tangent lengths is equal to the disk-director ratio q of the configuration.*

Equating (10) and (12) and using (13) for noncircular conics we also deduce

$$D = \frac{d}{q^2}, \tag{14}$$

which relates distance D between directors with distance d between centers of the focal disks.

Possible pairs of bifocal disks. Consider a fixed ellipse and its family of focal disks in Figure 4(a). Any two disks in this family can serve as a pair of focal disks. If the ellipse has eccentricity e, major axis of length $2a$, and distance $2f$ between foci, there are infinitely many pairs of positive constants c and d with ratio $d/c = e$ (because $q = e$) subject to the constraints $c \leq 2a$ and $d \leq 2f$, the same restrictions imposed by the inscribed spheres in Figure 12. If one permissible value of c or d is chosen, the other is determined by the relation $d/c = e$, and each such pair corresponds to a pair of focal disks. In Figure 4(a), the solid line joining the foci is the locus of centers of the focal disks.

For a fixed parabola ($q = e = 1$) there are infinitely many choices of positive constants c and d with $c = d$ (with no constraints), hence infinitely many pairs of focal disks (overlapping as well as nonoverlapping) with centers at distance d apart and with constant sum or absolute difference c of tangent lengths. Examples are shown in Figure 4(b).

Figure 5 shows that there is more than one way for pairs of focal disks to relate to a hyperbola. For normal configurations as in Figure 5(a), with $q = e = d/c$, one disk can be chosen inside each branch, or both disks can be chosen inside the same branch to form a pair of focal disks. Disks inside the same branch have no further constraints on c or d, but disks inside different branches require $d > 2f$, $c > 2a$. For abnormal configurations as in Figure 5(c), any two disks tangent to both branches of the hyperbola can be chosen as a pair of focal disks with no constraints on c or d except $d/c = q$, where now $q = \varepsilon$, the conjugate eccentricity of the hyperbola.

Tandem motion. Relations (9)–(14) reveal interesting phenomena not otherwise immediately apparent. The inscribed spheres in Figure 15 can be moved in tandem, that is, with fixed distance s between their centers. Their radii will change. The pierced disks in the cutting plane will also move and, by (11), distance d between their centers remains fixed, so the disks also move in tandem in the plane of the conic, and their radii will also change. The radii of the terminators will also change but, because α is constant, (9) and (10) tell us that the length c of the generator joining the terminators and the distance h between planes of the terminators do not change. Because c does not change the bifocal disk property holds with the same value of c during the entire tandem motion. By (12), distance D between directors does not change.

6. BIFOCAL DISK THEOREM AND ITS CONVERSE.

Every conic, including a flipped hyperbola, can be obtained as a section of some twisted cylinder. In view of the foregoing remarks we have the following bifocal disk theorem:

Theorem 3. *For every conic and each permissible value c of constant sum or absolute difference of tangent lengths, there is an infinite family of pairs of bifocal disks satisfying the bifocal disk property for that constant c.*

We remind the reader that for abnormal hyperbolic configurations, the family of bifocal disks in Figure 5(c) is not realizable on a cone.

The following converse of Theorem 3 extends the classical bifocal property of central conics [**1**, p. 498] to *all* conics, including the parabola. It also tells us that conics are the *only* curves having the bifocal disk property. Therefore this property characterizes the conics.

Theorem 4. *Suppose we are given two coplanar disks with distance $d > 0$ between their centers. The locus of all points in that plane such that either the sum or the absolute difference of the lengths of the tangent segments from this point to the two disks is a positive constant c is a conic. Each disk is also a focal disk with a director having disk-director ratio $q = d/c$, regardless of the radii of the disks. The eccentricity of the conic is related to q as described in Theorem* 2(b).

Note. Before presenting the proof, we note that the locus problem described in Theorem 4 involves four parameters (the two radii of the given disks, distance d between their centers, and constant c). Nevertheless, our proof will reduce this to the locus problem of Theorem 2, which involves only three parameters: the radius of one disk, the distance to its director, and the disk-director ratio.

Proof. Place the center of one disk of radius r_1 at the origin and the center of the other disk of radius r_2 at $(d, 0)$, where $d > 0$. Denote the lengths of the tangent segments from a point (x, y) on the locus by t_1 and t_2, as in Figure 16. Point (x, y) is on the locus if and only if either $t_1 + t_2$ or $t_1 - t_2$ is either c or $-c$. This holds if and only if either

$$(c - (t_1 + t_2))(c - (t_1 - t_2)) = 0$$

or

$$(c + (t_1 + t_2))(c + (t_1 - t_2)) = 0.$$

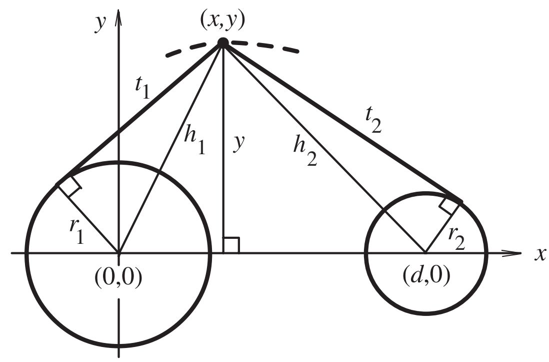


Figure 16. Diagram for the proof of Theorem 4.

Expanding, this is equivalent to

$$c^2 \pm 2ct_1 + (t_1^2 - t_2^2) = 0. \tag{15}$$

Right triangles with legs t_1 and t_2 reveal that $t_1^2 = h_1^2 - r_1^2$ and $t_2^2 = h_2^2 - r_2^2$, hence

$$t_1^2 - t_2^2 = h_1^2 - h_2^2 + r_2^2 - r_1^2. \tag{16}$$

But h_1 and h_2 are also hypotenuses of two right triangles with altitude y, so

$$h_1^2 = x^2 + y^2, \qquad h_2^2 = (x-d)^2 + y^2,$$

giving $h_1^2 - h_2^2 = 2\,dx - d^2$, which transforms (16) into

$$t_1^2 - t_2^2 = 2\,dx - d^2 + r_2^2 - r_1^2.$$

Use this in (15) and solve for t_1 to obtain

$$t_1 = \pm\left(\frac{d}{c}x + \frac{k}{2c}\right),$$

where

$$k = c^2 - d^2 + r_2^2 - r_1^2. \tag{17}$$

But $t_1 \geq 0$, hence

$$t_1 = \frac{d}{c}|x - \lambda_1| \tag{18}$$

where

$$\lambda_1 = -\frac{k}{2d}. \tag{19}$$

At this stage the locus problem has been reduced to that in Theorem 2(a). Comparing (18) with the defining property (2) and applying Theorem 2(a) we see that the locus is a conic. The disk of radius r_1 is also a focal disk having the line $x = \lambda_1$ as its inherited director and with disk-director ratio $q = d/c$. The Cartesian equation of the conic is given by (3) with $\lambda = \lambda_1$ and $r = r_1$. The eccentricity of the conic is described in terms of q by Theorem 2(b).

By an argument similar to that leading to (18), it is easy to verify that the disk of radius r_2 is also a focal disk *for exactly the same conic*, with its own inherited director $x = \lambda_2$ and the same disk-director ratio $q = d/c$, where

$$\lambda_2 = \lambda_1 + \frac{d}{q^2}. \tag{20}$$

Moreover, (20) shows that the directors are parallel and the distance between them is d/q^2, in agreement with (14). The directors are always orthogonal to the line through the centers of the disks. Examples illustrating this theorem are given in Section 10.

■

Note. Theorem 4 assumes $d > 0$ and does not cover the case of concentric focal disks. But in this case it is easy to verify directly that the locus in question is a concentric circle containing these disks. In fact, each of the tangent distances t_1 and t_2 from that circle to the focal disks is constant, so the sum of tangent distances is equal to the constant $t_1 + t_2$ and the absolute difference is the constant $|t_1 - t_2|$. Also, Theorem 4 does not cover the case $c = 0$ (equal tangent lengths), which is easily analyzed directly and is illustrated in the last diagram in Figure 22.

The solution of the four-parameter problem in Theorem 4 leads to the same set of conics as does the solution of the three-parameter problem in Theorem 2. This means that as we vary the four parameters in Theorem 4 we obtain conics from the same set with repetitions. This is consistent with the converse result in Theorem 3, which provides an infinite family of pairs of bifocal disks for a given conic.

7. TWO NEW CHARACTERIZATIONS OF THE CONICS. The results obtained in Theorems 1 through 4 can be summarized into one theorem that provides the two new characterizations of the conics mentioned in Section 1.

Theorem 5. *The focal disk-director property is satisfied by all noncircular conics, and only by the noncircular conics. The bifocal disk property is satisfied by all the conics, and only by the conics.*

A central conic has two foci and two directrices, but a parabola has one focus and one directrix. By contrast, we have shown that every conic, including the parabola, has infinitely many pairs of bifocal disks, and every noncircular conic has infinitely many focal disk-director pairs.

PART 3: SUPPLEMENTARY RESULTS

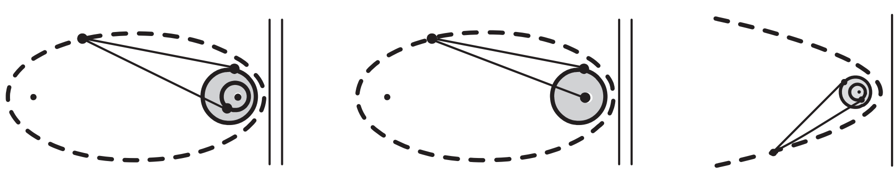
8. MORE ON DIRECTORS AND THE BIFOCAL DISK PROPERTY. If a director intersects the boundary of the focal disk at P, then in Figures 8(a) and 8(c), $PT = q\,PD = 0$ trivially, so P is on the conic. These intersections separate the conic into complementary portions, on one of which the sum of tangent lengths is constant, while on the other the absolute difference of tangent lengths is the same constant. Next we show how the directors identify these complementary portions.

Proposition 7. *Suppose we are given a conic having two focal disks with distance $d > 0$ between their centers. Let q be the disk-director ratio for each disk-director* *pair. Then on the portion of the conic between directors the sum of tangent lengths to the disks has the constant value $c = d/q$, and on the remaining portions the absolute difference of tangent lengths is the same constant.*

Proof. If P is on the conic with tangents to the focal disks of lengths t_1 and t_2, the distances from P to the corresponding directors are t_1/q and t_2/q. By (14), the distance between directors is d/q^2. Therefore, if P lies between the directors, then $t_1/q + t_2/q = d/q^2$, hence $t_1 + t_2 = d/q = c$. Otherwise, $|t_1/q - t_2/q| = d/q^2$, in which case $|t_1 - t_2| = d/q = c$. ■

Special Cases of Proposition 7. The classical bifocal properties of central conics are special cases of Proposition 7 for normal configurations. To see why, keep d fixed and let the focal disks shrink to the foci F_1 and F_2, so the directors become directrices. Every point P on an ellipse lies between its two directrices, so by Proposition 7 the sum $PF_1 + PF_2$ is constant. Points on a hyperbola cannot lie between the two directrices, and Proposition 7 tells us that the absolute difference $|PF_1 - PF_2|$ is constant on the entire hyperbola.

What happens if both disks are very near to one focus? Figure 17 shows examples with one focal disk inside another. In Figure 17(b), one of the disks is itself a focus. In these examples no portion of the conic is between the directors, revealing the surprising fact that the difference of tangent lengths can be constant *everywhere* on an ellipse and also on a parabola. This resembles the classical bifocal property of a hyperbola.



(a) (b) (c)

Figure 17. The difference of tangent lengths is constant everywhere on the ellipse in (a) and (b), and also everywhere on the parabola in (c).

9. MORE ON FOCAL DISKS AND DIRECTORS.

Next we address the following problem related to the families of focal disks in Theorems 1 and 3:

> *For a given noncircular conic, determine the center and radius of a possible focal disk.*

This problem can be solved by using the formulas in the proof of Theorem 2(b) to obtain, for each type of conic, a relation between the radius r and center of a focal disk. We omit algebraic details and give only the geometric interpretation of the relation, shown as a simple construction in Figure 18. For an ellipse of eccentricity e, horizontal dilation by the factor q $(= e)$ produces an auxiliary ellipse (shown dashed in Figure 18(a)) with its vertices at the foci of the given ellipse. On the major axis of the auxiliary ellipse, choose any point as center of the focal disk. Then the radius of the focal disk is the ordinate of the auxiliary ellipse, as shown in Figure 18(a). Horizontal dilation of any hyperbola in Figure 5 by a factor q produces an analogous auxiliary hyperbola (not shown) for constructing a focal disk. For a parabola the dashed auxiliary parabola in Figure 18(b) is obtained by translating the given parabola so the translated vertex matches the focus of the given parabola.

Figure 18. Auxiliary (dashed) conic used to construct a focal disk, for (a) an ellipse, and (b) a parabola.

How do we locate the director for a given focal disk?

This is answered by (14) which can be interpreted as the following:

Shifting Principle. *If one focal disk of a family is shifted through a distance d to another, its director will be shifted in the same direction by the distance d/q^2.*

In the abnormal case in Figure 5(c) we can start with the smallest disk whose director passes through its center, and shift it by distance d to another disk of the family. The director is then shifted in the same direction by d/q^2. For the families in Figures 4(a), 4(b), and 5(a), we can start with a focus as one focal disk, and shift it by distance d to another disk of the family. The directrix for that focus is then shifted by the distance d/q^2 to locate the director of the other disk.

10. EXAMPLES OF CONICS WITH FIXED BIFOCAL DISKS. This section fixes the focal disks and provides a series of snapshots showing what happens qualitatively to the conics as c varies. In Figures 19–22 we take $r_1 = 2$, $r_2 = 1$, $d = 6$ and let c decrease from $c = 10$ to $c = 0$. The arrows indicate the positions of the directors. Between directors the *sum* of tangent lengths to the disks is constant, as indicated by the solid curves. On the remaining portion of the conic the *absolute difference* of tangent lengths is constant, as indicated by the dashed curves. If c is large, the ratio d/c is close to zero, and the conic resembles a circle of radius $c/2$. As c decreases, the ratio d/c increases and the ellipse becomes more and more elongated (Figure 19). When $c = d$, the ellipse suddenly changes to a parabola, and then to a hyperbola as c continues to decrease to 0, as shown in Figures 20 through 22.

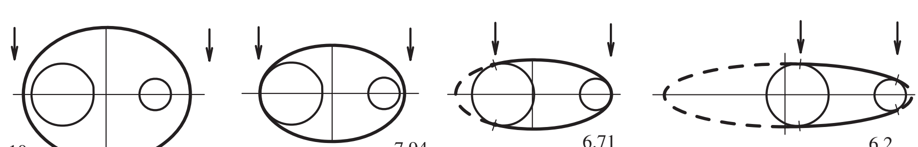


Figure 19. Ellipses become more elongated as c decreases from 10 toward 6.

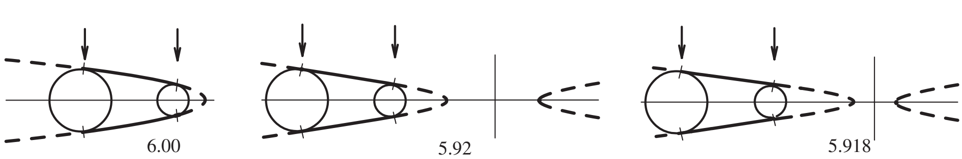


Figure 20. Conic becomes a parabola when $c = 6$, and a hyperbola when $c < 6$.

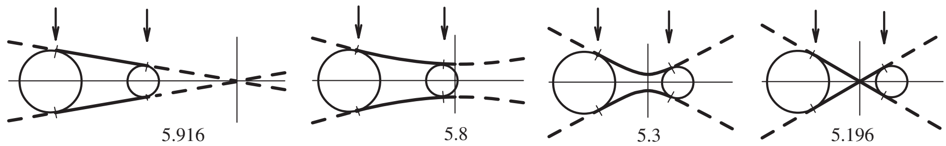


Figure 21. The hyperbola degenerates to a pair of lines when $c = \sqrt{d^2 - (r_1 - r_2)^2}$. As c decreases further the hyperbola flips and opens up and down instead of right and left, until the next degeneration occurs at $c = \sqrt{d^2 - (r_1 + r_2)^2}$. This interval for c represents abnormal configurations not realizable on a cone, but requiring a twisted cylinder.

When one disk is a point (a focus), the two extreme degenerate cases in Figure 21 coincide, and the abnormal configurations between them disappear. In this case, all possible configurations can be realized as sections of a cone, and the twisted cylinder is not needed. It is important to note that in this sequence of snapshots the disk-director ratio q increases monotonically, but the eccentricity e is not monotonic. The eccentricity jumps up when the hyperbola flips in Figure 21, and then jumps down when the hyperbola flips back in Figure 22. This is consistent with the fact that $e = q$ in all normal cases, and $e = q/\sqrt{q^2 - 1}$ in abnormal cases.

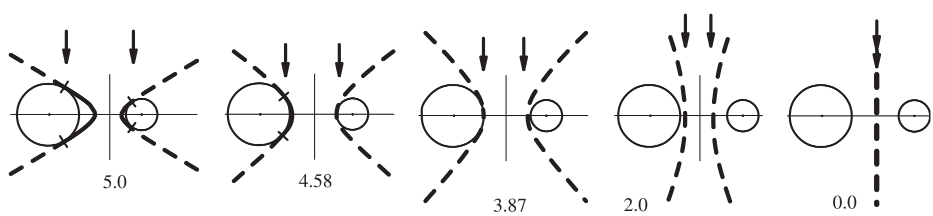


Figure 22. For $c < \sqrt{d^2 - (r_1 + r_2)^2}$ the hyperbola flips back with each branch tangent to one of the disks, until $c < \sqrt{(d - r_1)^2 - r_2^2}$, when both disks detach. As c approaches 0, both branches tend to one vertical line, on which the lengths of the tangents to the two disks are equal: $t_1 = t_2$.

11. CONCLUDING REMARKS. To the best of our knowledge, no one previously used cross sections of twisted cylinders to completely characterize the conic sections. Dandelin [**3**] used the hyperboloid of revolution to give new proofs of the Theorem of Pascal on hexagons inscribed in conic sections, and of the Theorem of Brianchon on hexagons circumscribing conic sections. As preparation he showed that every conic is a cross section of a hyperboloid cut by a plane, by inscribing spheres whose points of tangency with the cutting plane represent the foci of the conic, as he had done earlier on a cone [**1**, p. 498]. But in his treatment of the hyperboloid (and of the cone) he did not pierce the inclined plane with these spheres as we have done to produce focal disks and directors.

We are grateful to referees of an earlier version of this paper, who pointed out that some of our results were anticipated by Salmon [**6**, p. 241 and p. 263] and by Ferguson [**4**]. Although both Salmon and Ferguson briefly outline what we have called the focal disk-director property and the bifocal disk property of conics, their treatments are cursory. Neither shows, as we have done, that conics are completely characterized by the focal disk-director property or the bifocal disk property. In particular, neither proves that every conic has the bifocal disk property, neither relates focal disks and directors to twisted cylinders, and neither mentions infinite families of focal disks such as those described in Theorems 1 and 3.

Finally, we have pointed out geometric reasons why the twisted cylinder is more appropriate than the cone for studying conic sections. These are reinforced by analytic geometry. In Section 4 we found a standard Cartesian equation (3) for any locus satisfying the focal disk-director property. In some cases, the resulting locus cannot be realized on a cone. For example, the equation $y^2 = r^2$ represents two parallel lines. Also, flipped hyperbolas cannot appear on the same cone, but they do appear as sections of one twisted cylinder. Thus, the twisted cylinder is a more natural platform than the cone for investigating conics.

REFERENCES

1. T. M. Apostol, *Calculus,* vol. 1, 2nd ed., John Wiley, New York, 1967.
2. T. M. Apostol and M. A. Mnatsakanian, Proof without words: Surprising property of hyperbolas, *Math. Mag.* **79** (2006) 339.
3. G. P. Dandelin, Mémoire sur l'hyperboloide de révolution et sur les hexagones de Pascal et de M. Brianchon, *Nouveaux mémoires de l'Académie royale des sciences et belles-lettres de Bruxelles* **3** (1826) 1–14. English translation available at `http://www.math.ubc.ca/~cass/dandelin.pdf`.
4. D. F. Ferguson, Theorems on conics, *Mathematical Gazette* **31** (1947) 47–49.
5. M. A. Mnatsakanian, Twisting a cylinder, Interactive animation to form various twisted cylinders, 2000, available at `http://www.its.caltech.edu/~mamikon/TwistCyl.html`.
6. G. Salmon, *A Treatise on Conic Sections*, 6th ed., Chelsea, New York, 1960.

TOM M. APOSTOL joined the Caltech mathematics faculty in 1950 and became professor emeritus in 1992. He is director of Project MATHEMATICS! (`http://www.projectmathematics.com`) an award-winning series of videos he initiated in 1987. His long career in mathematics is described in the September 1997 issue of *The College Mathematics Journal*. He is currently working with colleague Mamikon Mnatsakanian to produce materials demonstrating Mamikon's innovative and exciting approach to mathematics.
California Institute of Technology, 253-37 Caltech, Pasadena, CA 91125
apostol@caltech.edu

MAMIKON A. MNATSAKANIAN received a Ph.D. in physics in 1969 from Yerevan University, where he became professor of astrophysics. As an undergraduate he began developing innovative geometric methods for solving many calculus problems by a dynamic and visual approach that makes no use of formulas. He is currently working with Tom Apostol under the auspices of Project MATHEMATICS! to present his methods in a multimedia format.
California Institute of Technology, 253-37 Caltech, Pasadena, CA 91125
mamikon@caltech.edu