Unwrapping Curves from Cylinders and Cones

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1. INTRODUCTION. In his delightful book *Mathematical Snapshots*, Steinhaus [**1**] describes the simple, engaging construction illustrated in Figure 1. Wrap a piece of paper around a cylindrical candle, and cut it obliquely with a knife. The cross section is an ellipse, which becomes a sinusoidal curve when unwrapped.

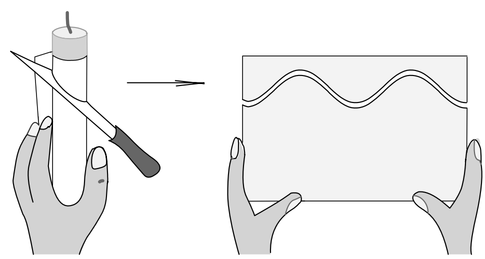


Figure 1. An elliptical cross section of a cylinder becomes sinusoidal when unwrapped.

The same idea can be demonstrated with a safer instrument. Take a cylindrical paint roller, dip it at an angle in a container of paint or water color, and roll it on a flat surface. The roller prints a sinusoidal wave pattern as shown in Figure 2.

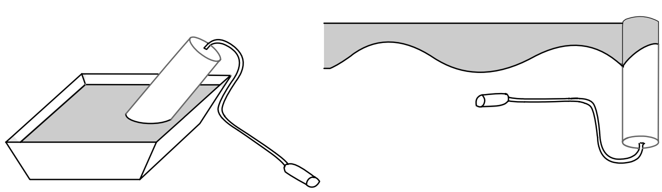


Figure 2. A paint roller used to print sinusoidal waves on a flat surface.

Now imagine the elliptical cross section replaced by any curve lying on the surface of a right circular cylinder. *What happens to this curve when the cylinder is unwrapped?*

Consider also the inverse problem, which you can experiment with by yourself: Start with a plane curve (line, circle, parabola, sine curve, etc.) drawn with a felt pen on a rectangular sheet of transparent plastic, and roll the sheet into cylinders of different radii. *What shapes does the curve take on these cylinders? How do they appear when viewed from different directions?* A few trials reveal an enormous number of possibilities, even for the simple case of a circle.

This paper formulates these somewhat vague questions more precisely, in terms of equations, and shows that they can be answered with surprisingly simple two-dimensional geometric transformations, *even when the cylinder is not circular*. For a circular cylinder, a sinusoidal influence is always present, as exhibited in Figures

1 and 2. And we demonstrate, through examples, applications to diverse fields such as descriptive geometry, computer graphics, printing, sheet metal construction, and educational hands-on activities.

Starting with section 9, we also investigate what happens to a space curve unwrapped from the lateral surface of a right circular cone. For example, a plane cuts the cone along a conic section, and we can analyze the shape of the corresponding unwrapped conic. This leads to a remarkable family of periodic plane curves that apparently have not been previously investigated. The family is described by a polar equation resembling that for a conic section. We call members of this family *generalized conics*; limiting cases include all ordinary conic sections, as well as the sinusoidal curves mentioned earlier.

2. UNWRAPPING AN ELLIPSE FROM A CIRCULAR CYLINDER. Before turning to the general problem, let's analyze the foregoing sinusoidal construction. Cut a right circular cylinder of radius r by a plane through a diameter of its base at angle of inclination β, where $0 < \beta < \pi/2$. The example in Figure 3a shows part of the elliptical cross section and a wedge cut from the cylinder. A vertical cutting plane parallel to the major axis of the ellipse intersects the wedge along a right triangle T (shown shaded) with base angle β.

When the lateral surface of the cylinder is unwrapped onto a plane, the circular base unfolds along a line we call the x-axis. Here x is the length of the circular arc measured from point A at the extremity of the base diameter to point B at the base of triangle T, as shown in Figure 3a. The base of T has length $r\sin(x/r)$, and its height is $h\sin(x/r)$ where $h = r\tan\beta$, so the unwrapped curve is the graph of the function

$$u(x) = h\sin\frac{x}{r},$$

representing a sinusoidal curve with period $2\pi r$ and amplitude h. For fixed r the amplitude h increases with β.

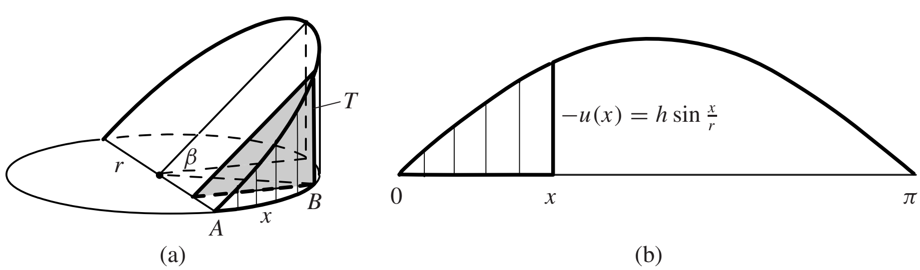


Figure 3. The circular arc AB of length x in (a) unwraps onto the line segment $[0, x]$ in (b). The altitude of triangle T unwraps onto the height $u(x)$.

An adjacent arch below the x-axis comes from unwrapping the second symmetrically located wedge. Volume calculations of cylindrical wedges like these were considered by Archimedes and are analyzed in more detail in [2], where unwrapping of a cylinder is also used to deduce the quadrature of a sine curve without integral calculus.

3. CURVE OF INTERSECTION OF TWO CYLINDERS. A *cylinder* is any surface generated or swept out by a straight line moving along a plane curve and remaining parallel to a given line. The curve is called a *directrix* of the cylinder, and the moving line that sweeps out the cylinder is called a *generator.* The directrix is not unique because any plane cuts a given cylinder along a plane curve that can serve as directrix. When the cutting plane is perpendicular to the generators we call the directrix a *profile* of the cylinder.

A curve in the xy-plane has an implicit Cartesian equation of the form

$$m(x, y) = 0.$$

In xyz-space this equation describes a cylinder having this profile, with generators parallel to the z-axis. Similarly, an equation of the form $p(x, z) = 0$ (with y missing) describes a cylinder with generators parallel to the y-axis, whereas one of the form $q(y, z) = 0$ (with x missing) describes a cylinder with generators parallel to the x-axis.

Start with a vertical cylinder in xyz-space with equation $m(x, y) = 0$, which we call the *main cylinder*, and locate the coordinate axes so that the profile passes through the origin, and the z-axis lies along a generator. Intersect the main cylinder with a horizontal cylinder $p(x, z) = 0$, which we refer to as the *cutting cylinder*. Their curve of intersection C is the set of points (x, y, z) that satisfy both $m(x, y) = 0$ and $p(x, z) = 0$. Let C_p denote the profile of the cutting cylinder, which shows what C looks like when viewed along the generators of the cutting cylinder (see Figure 4a). We call the xz-plane the *viewing plane* and the equation $p(x, z) = 0$ the *profile equation*.

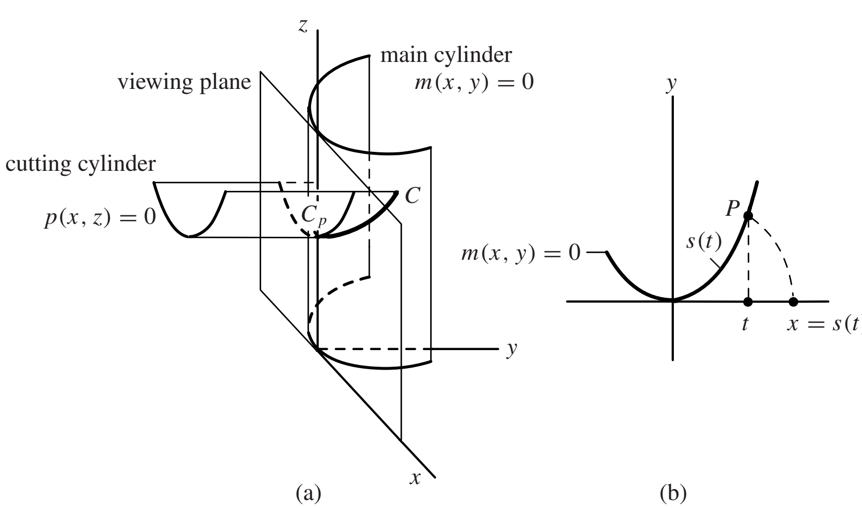


Figure 4. (a) The main cylinder and an orthogonal cutting cylinder intersect along C. (b) Point P on the horizontal profile projects onto $(t, 0)$ and unwraps onto $(s(t), 0)$.

4. UNWRAPPING A CURVE FROM ANY CYLINDER. Now unwrap the main cylinder onto the xz-plane, which we describe as the *unwrapping plane*. A curve C on the cylinder is printed onto an unwrapped curve C_u in the xz-plane. It has an equation of the form $u(x, z) = 0$, its *unwrapping equation*, that we shall determine from the profile equations $m(x, y) = 0$ and $p(x, z) = 0$ that define C.

We use the fact that every cylinder is a developable surface, hence unwrapping preserves distances. In particular, *any arc of length s on the (horizontal) profile* $m(x, y) = 0$ *gets printed onto a line segment of the same length on the x axis*.

To formulate this in terms of equations, imagine each point P on the profile of the main cylinder described in terms of new parameters t and $s(t)$, where $s(t)$ is the arclength of the portion of the profile joining the origin to P, and t is the projection of that arc on the x-axis. Unwrapping the cylinder prints point P onto a point of the xz-plane with coordinates $(s(t), 0)$. Hence any other point on curve C at height z above P is printed onto the point $(s(t), z)$, where z satisfies the profile equation $p(t, z) = 0$. Consequently, in the unwrapping equation $u(x, z) = 0$, x and z are related as follows: $x = s(t)$ and z satisfies $p(t, z) = 0$. We plot t on the x-axis.

To express u directly in terms of p, we consider portions of C for which the function $x = s(t)$ has an inverse, so that t can be expressed in terms of x by the relation $t = s^{-1}(x)$. Under these conditions, we have the following theorem:

Theorem 1. *For a curve C as described, the profile equation $p(t, z) = 0$ for C_p and the unwrapping equation $u(x, z) = 0$ for C_u are related as follows:*

$$u(x, z) = p(s^{-1}(x), z) \tag{1}$$

$$p(t, z) = u(s(t), z). \tag{2}$$

When curves C_p and C_u are described by explicit equations, we use the same letters p (for profile) and u (for unwrapping), and write $z = p(t)$ and $z = u(x)$, respectively. In this case (1) and (2) become

$$u(x) = p(s^{-1}(x)), \qquad p(t) = u(s(t)).$$

In other words, to obtain the profile function $p(t)$ from $u(x)$, simply replace the argument x with the arclength $s(t)$. Conversely, to obtain the unwrapping function $u(x)$ from $p(t)$, simply replace the argument t with the inverse $s^{-1}(x)$. The following special case is worth noting. It is illustrated in Examples 1 and 2.

Corollary. *For the linear unwrapping function $u(x) = x$, the profile function is the arclength function: $p(t) = s(t)$. And for the linear profile function $p(t) = t$, the unwrapping function is the inverse of the same arclength function: $u(x) = s^{-1}(x)$.*

5. UNWRAPPING A CURVE FROM A CIRCULAR CYLINDER. Figure 5 illustrates these ideas when the main cylinder is a right circular cylinder of radius r. In Figure 5c, a circular arc of length $r\theta$ is unwrapped onto a segment of length x, and its horizontal projection has length $t = r \sin\theta$. Because $\theta = x/r$, the arclength function is $x = s(t) = r \arcsin(t/r)$, its inverse is $t = s^{-1}(x) = r \sin(x/r)$, and Theorem 1 becomes:

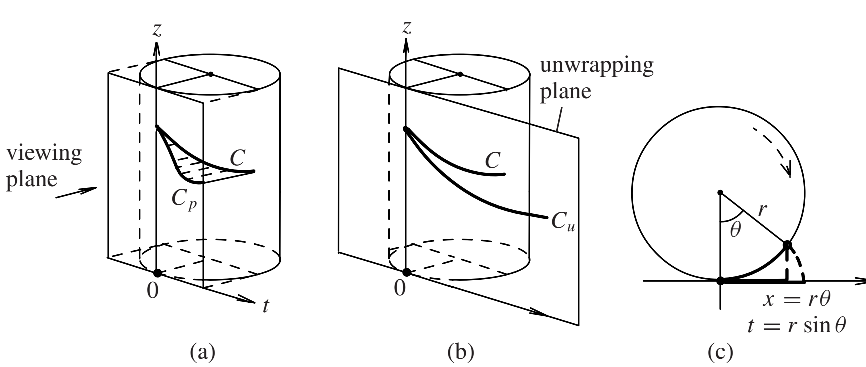


Figure 5. (a) Curve C on a circular cylinder and its horizontal profile C_p on a tangent viewing plane. (b) The unwrapped curve C_u obtained by rolling the cylinder along the unwrapping plane. (c) A point on a circle projects onto point $t = r\sin\theta$, but unwraps onto point $x = r\theta$.

Theorem 2. *On a right circular cylinder of radius r, let C be a curve defined by the profile C_p of a horizontal cutting cylinder, and let C_u denote its unwrapped image. Then the profile equation $p(t, z) = 0$ for C_p and the unwrapping equation $u(x, z) = 0$ for C_u are related as follows:*

$$u(x, z) = p\left(r \sin \frac{x}{r}, z\right), \tag{3}$$

$$p(t, z) = u\left(r \arcsin \frac{t}{r}, z\right). \tag{4}$$

This shows that the sine function is always present when an arbitrary curve is unwrapped from a right circular cylinder onto a plane. When C_p and C_u are described by explicit equations, say $z = p(t)$ and $z = u(x)$, then (3) and (4) become

$$u(x) = p\left(r \sin \frac{x}{r}\right), \tag{5}$$

$$p(t) = u\left(r \arcsin \frac{t}{r}\right). \tag{6}$$

When $r = 1$, $\arcsin t = s(t)$, the length of the circular arc whose sine is t.

Now we apply Theorem 2 to some simple examples.

Example 1 (Linear profile function $p(t) = ct$). In this case, the cutting cylinder is a plane through the line $z = ct$, where c is constant. From (5) we find that the unwrapping function is $u(x) = cr \sin(x/r)$, whose graph is a sinusoidal curve with period $2\pi r$.

If the cutting plane is inclined at an angle β with a horizontal diameter of the cylinder, then $c = \tan \beta$ and the unwrapping function is $u(x) = h \sin(x/r)$, where $h = r \tan \beta$, in agreement with the result obtained earlier by analyzing Figure 3. The shape of the cross section curve C itself depends on the direction from which it is viewed. When viewed along the edge of the cutting plane we see the profile C_p as a line segment. In a later example we show that when viewed from any direction the cross section cut by a plane is, as expected, always an ellipse (possibly degenerate).

Example 2 (Linear unwrapping function $u(x) = cx$). This example explains what happens when a straight line on a transparency is rolled onto a cylinder of radius r. The corresponding profile function obtained from (6) is

$$p(t) = cr \arcsin \frac{t}{r}.$$

Because distances are preserved when the cylinder is unwrapped, a line segment on the unwrapped cylinder (the shortest path between its endpoints) becomes a geodesic arc (the shortest path) on the cylinder, no matter how tightly it is rolled. In other words, on any right circular cylinder the profile of a geodesic arc is part of an arcsine curve. Figure 6a shows the line $u(x) = x$ in the unwrapping plane, and Figures 6b–e show the profile of the geodesic arc on cylinders of decreasing radii. Again we see sinusoidal curves, but they are flipped sideways, as predicted by the corollary to Theorem 1. The dashed curve in Figure 6d indicates the portion of the geodesic arc that lies on the "rear" of the cylinder. The two repeated curves in Figure 6e represent different branches of the arcsine function.

This suggests a simple educational hands-on activity that can engage young students while they learn that a geodesic on a circular cylinder is always part of a circular helix. Use a felt pen to draw a line segment on a transparency, roll it into a circular cylinder, view the profile in various directions, and watch the sine waves change shape as the radius of the cylinder varies.

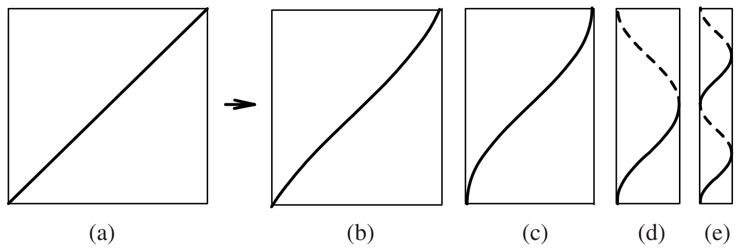


Figure 6. A line segment (a) wraps onto a geodesic. In (b)–(e) geodesic profiles (arcsine curves) are shown on cylinders of decreasing radii. The dashed curves in (d) and (e) are on the rear half of the cylinder.

Example 3 (Parabolic cutting cylinder). Figure 7 shows a quadratic profile equation $p(t) = ct^2$ for some constant $c > 0$. By (5) the corresponding unwrapping function is

$$u(x) = cr^2 \sin^2 \frac{x}{r}. \tag{7}$$

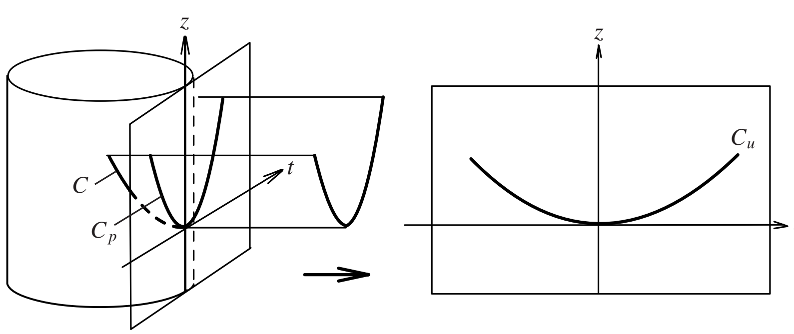


Figure 7. Curve C cut by a parabolic cylinder, with profile C_p and unwrapped curve C_u.

Examples of this type have practical applications. To illustrate, take a rectangular piece of sheet metal cut along the curve described by (7), and roll it to form a circular cylinder of radius r. The curve C cut out on the resulting cylinder indicates exactly where it will intersect a parabolic "gutter" having profile $p(t) = ct^2$.

Example 4 (Wrapping a circle onto a cylinder). On a sheet of transparent plastic, draw a unit circle, and roll the sheet into a circular main cylinder. What does the wrapped circle look like when viewed from the side? The circle wraps onto a curve C and we want its horizontal profile C_p. The upper half of the unwrapped unit circle can be described by the unwrapping function

$$u_+(x) = \sqrt{1 - x^2},$$

and the lower half by $u_-(x) = -\sqrt{1-x^2}$, shown dotted in Figure 8a. Both halves can be described by the implicit equation $u^2(x) = 1 - x^2$. From (5) we see that the corresponding profile functions are $p_\pm(t) = u_\pm(r \arcsin(t/r))$, both of which can be described by the implicit equation

$$p^2(t) = 1 - \left(r \arcsin \frac{t}{r}\right)^2.$$

They depend on the radius of the main cylinder. In Figures 8b–f the cylinder is turned (for ease in displaying) so that its axis is horizontal, and the corresponding graph of $p_+(t)$ is shown for various values of r. The flipped graph of each $p_-(t)$ (not shown) is the mirror image reflection through the horizontal dashed line. In Figures 8c–f the dashed curves lie on the rear half of the cylinder and are hidden from view.

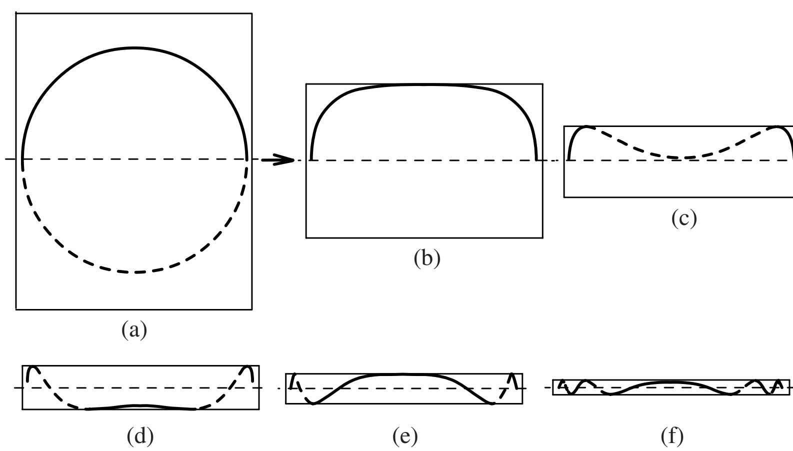


Figure 8. Rolling a circle on a transparency onto cylinders with decreasing radii.

6. ROTATING THE MAIN CYLINDER. On a circular cylinder of radius r, take a curve C with explicit profile function $z = p(t)$. Rotate the cylinder through an angle α about its axis, but keep the viewing plane fixed. The profile function of the rotated curve on the viewing plane depends on α, and we denote its ordinates by z_α. The next theorem describes z_α in terms of p.

Theorem 3. *On a cylinder of radius r, take a curve C with profile function $z = p(t)$ on the viewing plane. If the cylinder is rotated about its axis through an angle α, the rotated curve on the same viewing plane has profile function*

$$z_\alpha = p\big(t \cos\alpha + \sqrt{r^2 - t^2} \sin\alpha\big). \tag{8}$$

Proof. Rotation of the cylinder through an angle α is equivalent to shifting the arclength $x = r\theta$ by an amount $r\alpha$. Therefore, if the unwrapping function of C is $u(x)$, rotation of the cylinder through an angle α (measured clockwise when viewed from above) replaces x with $x + r\alpha$, and the unwrapping equation of rotated C becomes

$$z_\alpha = u(x + r\alpha) = p\left(r \sin \frac{x + r\alpha}{r}\right)$$

by (5). But

$$r\sin\left(\frac{x}{r}+\alpha\right) = r\sin\frac{x}{r}\cos\alpha + r\cos\frac{x}{r}\sin\alpha.$$

In terms of $t = r\sin(x/r)$, we have $r\cos(x/r) = \sqrt{r^2 - t^2}$, and the foregoing equation for z_α becomes

$$z_\alpha = p\left(r\sin\frac{x + r\alpha}{r}\right) = p\big(t\cos\alpha + \sqrt{r^2 - t^2}\sin\alpha\big),$$

which proves (8). Note that $z_0 = p(t)$. ■

It is not surprising that the combination $t\cos\alpha + \sqrt{r^2 - t^2}\sin\alpha$ in (8) resembles the right-hand side of the equation $x' = x\cos\alpha + y\sin\alpha$ for changing coordinates from an xy-system to an $x'y'$-system by rotation of axes through an angle α.

We leave it to the reader to formulate the result corresponding to (8) when the profile is given in implicit form $p(z, t) = 0$.

Corollary (Perpendicular view). *When $\alpha = \pi/2$, we get the profile function*

$$z_{\pi/2} = p\big(\sqrt{r^2 - t^2}\big).$$

Example 5 (Rotated view of a slanted plane cut). If $p(t) = t$, which corresponds to cutting the original cylinder by a plane inclined at 45°, relation (8) translates to

$$z_\alpha = t\cos\alpha + \sqrt{r^2 - t^2}\sin\alpha,$$

which implies that

$$z_\alpha^2 - 2z_\alpha t\cos\alpha + t^2 = r^2\sin^2\alpha.$$

As expected, this represents an ellipse (possibly degenerate) in the tz_α-plane. Examples are shown in Figure 9. When $\alpha = 0$ the profile is a line segment with equation $z = t$, and when $\alpha = \pi/2$ it is the circle described by $z^2 + t^2 = r^2$.

Example 6 (Rotated view of a parabolic cut). In this case $p(t) = ct^2$ and (8) takes the form

$$z_\alpha = c\big(t\cos\alpha + \sqrt{r^2 - t^2}\sin\alpha\big)^2.$$

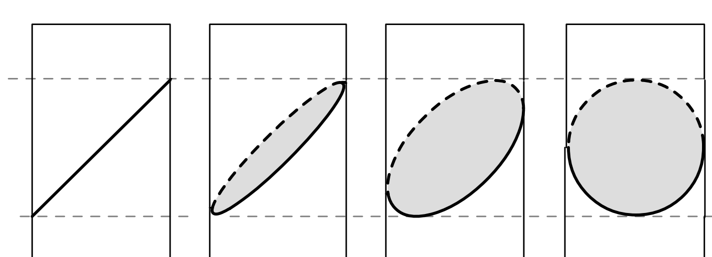


Figure 9. Various profiles of a rotating inclined ellipse. The dashed portions lie on the rear half.

For $c = 1$, Figure 10 shows the profile curve C_p for various values of α. Surprisingly, when the cylinder is rotated through a right angle, the profile is the mirror image of the original parabola reflected through the line $z = r^2/2$.

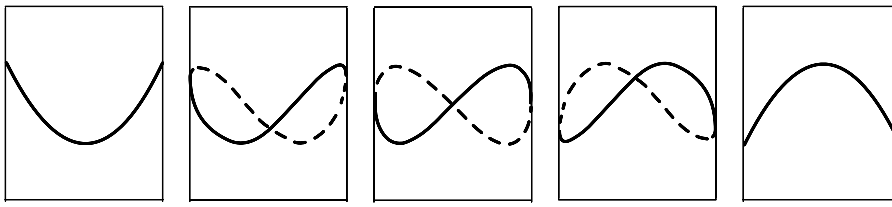


Figure 10. Rotated views of a parabolic intersection are depicted. In a perpendicular direction we see the original parabola flipped upside down.

7. DRILLED CYLINDERS.

Drill a hole through the main cylinder of radius r with a circular cylindrical drill of radius a whose axis is perpendicular to the viewing plane and at distance d from the axis of the given cylinder, where $0 \le d \le r + a$. The edge of the hole is a curve C on the given cylinder that appears as part of a circle when viewed along the axis of the drill. A plane through the axis of the main cylinder parallel to the viewing plane divides the cylinder into two parts, a "front portion" and a mirror image "rear portion." Curve C also consists of two parts, one lying on the front portion and its mirror image on the rear portion. The two parts can be connected or disconnected, depending on the size and position of the hole. Also, the corresponding unwrapped curve C_u is symmetric about the vertical line $x = \pi r/2$.

Place the axis of the drill so it intersects the t-axis of the viewing plane orthogonally at $(d, 0)$. To find the unwrapping function $z = u(x)$ of C, first we find the profile function $z = p(t)$, then by (6) we have $u(x) = p(r \sin(x/r))$. When $r = 1$, we have $u(x) = p(\sin x)$.

To determine $p(t)$, note that each projected point $(t, p(t))$ in the viewing plane lies on a circle of radius a with center at $(d, 0)$, so $p(t)^2 + (d - t)^2 = a^2$. Hence the upper and lower halves of the circular hole have profile functions that satisfy

$$p^2(t) = a^2 - (d - t)^2. \tag{9}$$

The corresponding unwrapping functions satisfy

$$u^2(x) = a^2 - \left(d - r \sin \frac{x}{r}\right)^2. \tag{10}$$

The following examples display interesting families of unwrapped curves obtained when a, d, and r are treated as parameters.

Example 7 (Drill of same radius as main cylinder; variable distance *d*). Take $a = r = 1$ in (10), and let distance d decrease from 2 to 0. When $d = 2$, the drill is tangent to the main cylinder at one point, which unwraps onto the single point $(\pi/2, 0)$. For $d = 1$, the unwrapping equation is

$$u^2(x) = 2 \sin x - \sin^2 x.$$

As d decreases, the hole changes shape, reaching its maximum size when $d = 0$, at which stage the drill's axis passes through the axis of the main cylinder and

$$u^2(x) = 1 - \sin^2 x = \cos^2 x.$$

Figure 11 shows the upper half of the unwrapped curve for a few decreasing values of d. Each curve shown in Figure 11 includes the unwrapped symmetric image that comes from the rear portion of the main cylinder, with vertical axis of symmetry $x = \pi/2$. In each case the lower half (not shown) can be obtained by reflecting the curve through the x-axis. Incidentally, the graph of $z = |\cos x|$, together with its reflection $z = -|\cos x|$ $(|x| \leq \pi/2)$, represents the unwrapped intersection of two perpendicular cylinders of unit radius. In this case the intersection itself is an ellipse (see [**2**]).

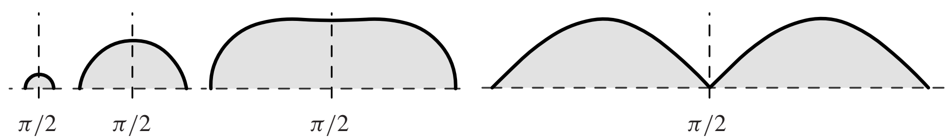


Figure 11. The unwrapped curve (upper half) is obtained by drilling a hole of unit radius through different parts of the main cylinder of the same radius. Each is symmetric about the line $x = \pi/2$. The lower half (not shown) is the reflection of the upper half through the x-axis.

Example 8 (Centered drill of variable radius). If $r = 1$ and $d = 0$, the hole is centered on the axis of the main cylinder, and (10) simplifies to $u^2(x) = a^2 - \sin^2 x$, which represents a family of unwrapped curves depending on the radius a of the hole. In this example the geometry and the equation itself show that each unwrapped curve is symmetric about the line $x = 0$. Figure 12 shows a few members of the family, $z = \sqrt{a^2 - \sin^2 x}$, for increasing values of a, from very small radius to very large radius. The case $a = 1$ gives the third curve, whose equation is $z = |\cos x|$, also shown in Figure 11.

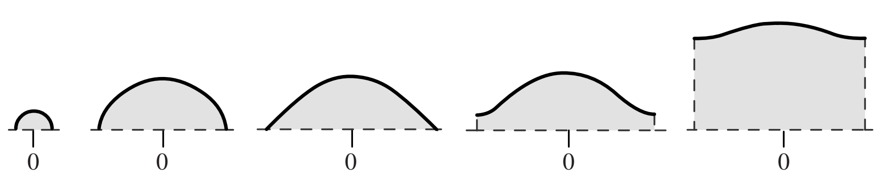


Figure 12. The unwrapped image of the upper half of a hole of variable radius drilled through the axis of the main cylinder is shown. Each is symmetric about the line $x = 0$. The lower half (not shown) is the reflection of the upper half through the x-axis.

Example 9 (Rotated view of a drilled circular hole). We return to (9), which describes the projected view of the hole obtained by drilling a hole through the main cylinder of radius r with a drill of radius a whose axis is perpendicular to the viewing plane and at distance d from the axis of the given cylinder, where $0 \leq d \leq r + a$. What is the shape of the hole as projected on the viewing plane after the main cylinder has been rotated through a right angle? Applying the general rotation formula in (8) with $\alpha = \pi/2$ and $p(t)$ as given in (9), we find that the upper and lower halves of the hole in the rotated cylinder satisfy the equation

$$z^2 = a^2 - \left(d - \sqrt{r^2 - t^2}\right)^2. \tag{11}$$

This can be written as $z^2 = a^2 - d^2 - r^2 + t^2 + 2d\sqrt{r^2 - t^2}$, or

$$(z^2 - a^2 + d^2 + r^2 - t^2)^2 = 4d^2(r^2 - t^2),$$

a Cartesian equation of degree 4 in both t and z.

When $d = r + a$, the drill is tangent to the main cylinder. As d decreases towards 0 the projection of the rotated hole changes its appearance. When $d = 0$, the drill passes through the axis of the main cylinder and the Cartesian equation reduces to $t^2 - z^2 = r^2 - a^2$, which represents an equilateral hyperbola if $a \neq r$. The hyperbola has a horizontal axis if $a < r$ and a vertical axis if $a > r$. If $a = r$, the radius of the drill is the same as that of the main cylinder and the rotated projected curves are the pair of lines given by $z = \pm t$.

Figure 13 shows how the projection changes its appearance when $r = a = 1$ and the axis of the drill moves toward the axis of the main cylinder. In Figures 13a–d it has the appearance of an expanding oval. In Figure 13e it becomes nonconvex, then gradually deforms to resemble hyperbolas (Figure 13f). Finally the two axes intersect when $d = 0$, when the projection becomes a pair of lines, a degenerate hyperbola (Figure 13g).

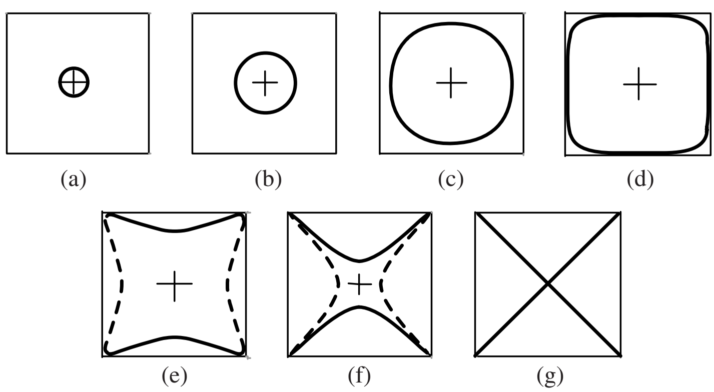


Figure 13. Profiles of the hole made by a drill of radius equal to that of the main cylinder, as the axis of the drill moves toward the axis of the main cylinder, viewed from a direction perpendicular to the axis of the drill.

Figure 14 shows a corresponding sequence when $r = 1$ and $a = 1/2$. When $d = 0$ an equilateral hyperbola suddenly appears. When $a > 1$ the projections are like those in Figure 14, but turned sideways by 90°.

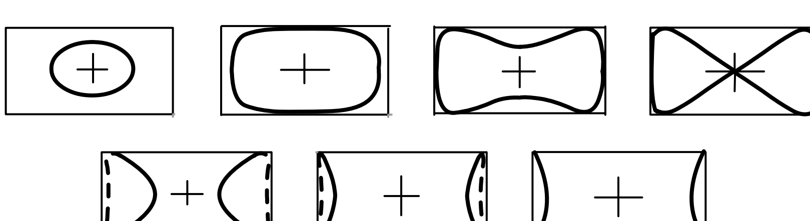


Figure 14. Profiles of the hole made by a drill of radius half that of the main cylinder, as the axis of the drill moves toward the axis of the main cylinder, viewed from a direction perpendicular to the axis of the drill.

8. TILTED CUTTING CYLINDER. Up to now we have studied the profile of a curve C cut from the main cylinder by an orthogonal cutting cylinder. In descriptive geometry and in applications to sheet metal work the cutting cylinder is not always orthogonal to the main cylinder but may be tilted at an angle β. For such applications we want to know what the unwrapped version C_u looks like so we can cut the unwrapped cylinder along this curve. To find C_u it suffices to determine the horizontal projection C_p, which is related to the profile of the slanted cutting cylinder. This relation is provided by the following theorem, whose proof is omitted:

Theorem 4. *Let C be the curve of intersection of a main cylinder with horizontal profile $y = m(t)$ that is cut by a cylinder tilted at an angle β, with profile function $z' = q(t)$. Then the horizontal projection $z = p(t)$ of C_p is related to the function $z' = q(t)$ by the equation*

$$q(t) = p(t)\cos\beta + m(t)\sin\beta. \tag{12}$$

Special cases.

(a) If $\beta = 0$, this gives $q(t) = p(t)$.

(b) If $\beta = \pi/2$, (12) becomes $z' = m(t)$. This is to be expected, because the viewing direction is along the axis of the main cylinder and every curve on the main cylinder appears as part of its profile.

(c) If $p(t) = 0$, (12) simplifies to $z' = m(t)\sin\beta$, a scaled version of the profile of the main cylinder. In particular, if the main cylinder has a circular profile, $m(t) = r - \sqrt{r^2 - t^2}$, then when $\beta \neq 0$ the relation $z' = m(t)\sin\beta$ can also be written as

$$\frac{t^2}{r^2} + \left(\frac{z' - r\sin\beta}{r\sin\beta}\right)^2 = 1.$$

This is the equation of an ellipse with center at $(0, r\sin\beta)$ in the tz'-plane and with semiaxes of lengths r and $r\sin\beta$. In this case, curve C is a circle of radius r on the main cylinder, and it appears as an ellipse when projected on a tilted plane.

(d) If the main cylinder has a circular profile $m(t) = r - \sqrt{r^2 - t^2}$, then as $r \to \infty$ the main cylinder becomes a plane, the quantity $r - \sqrt{r^2 - t^2} \to 0$, and (12) becomes $q(t) = p(t)\cos\beta$, as expected.

(e) When $\cos\beta \neq 0$, (12) can be solved for $p(t)$ to give

$$p(t) = q(t)\sec\beta - m(t)\tan\beta, \tag{13}$$

a linear combination of the two profile functions $q(t)$ and $m(t)$. In particular, if the cutting cylinder is a circular cylinder of radius a cutting a main circular cylinder of radius r at angle β, then (13) holds with $q(t) = \sqrt{a^2 - t^2}$ and $m(t) = r - \sqrt{r^2 - t^2}$.

Example 10 (Intersection of two circular cylinders). Now take the special case of (e) in which $a = r$. Let $z = p(t)$, and write (13) in the form

$$z + r\tan\beta = \sqrt{r^2 - t^2}(\sec\beta + \tan\beta)$$

or

$$\frac{t^2}{r^2} + \left(\frac{z + r\tan\beta}{r(\sec\beta + \tan\beta)} \right)^2 = 1.$$

This is the equation of an ellipse with center at $(0, -r\tan\beta)$ in the tz-plane and semi-axes of lengths r and $r(\sec\beta + \tan\beta)$. Because the projected curve C_p is an ellipse, we know that the unwrapped curve C_u will be sinusoidal.

Example 11 (Tilted view of a geodesic). Example 2 revealed that a geodesic on a right circular cylinder is a circular helix whose side view is a sine curve. On a circular cylinder of radius 1, the geodesic with unwrapping function $u(x) = cx$ has horizontal profile $p(t) = c \arcsin t$, and the main cylinder has circular profile $m(t) = 1 - \sqrt{1 - t^2}$. Hence (12) gives

$$q(t) = c(\arcsin t)\cos\beta + \left(1 - \sqrt{1 - t^2}\right)\sin\beta.$$

When $\tan\beta = c$, the helix is viewed along one of its tangents of constant slope, and this becomes

$$\frac{1}{\sin\beta} q(t) = \arcsin t + \left(1 - \sqrt{1 - t^2}\right).$$

The right member is easily shown to represent a cycloid, the path traced out by a point on the circumference of a circular disk that rolls along a straight line, so the profile $q(t)$ describes a cycloid dilated in the z' direction by the factor $\sin\beta$.

This can also be demonstrated physically with a flexible spring, such as the toy known as a “slinky.” By stretching the spring and viewing it from different directions you can see the helix change its appearance from sine curve to curtate cycloid, to cycloid, and to prolate cycloid.

9. UNWRAPPING CURVES FROM A RIGHT CIRCULAR CONE. The examples treated thus far involve curves unwrapped from a right circular cylinder. Now we start with a curve C lying on the surface of a cone, and consider questions of the following type: *What is the shape of the image of C when the cone is unwrapped, that is, tipped on a generator and rolled onto a plane, or onto another cone? How does curve C appear when viewed from different directions?*

All cones in this paper are right circular cones, and we analyze not only conic sections, but any curve lying on the surface of such a cone. We employ three simple geometric transformations: projecting the curve onto the *ceiling plane* (a plane orthogonal to the cone’s axis and passing through its vertex), scaling the ceiling projection radially from the vertex, and compressing the polar coordinate angle.

We formulate the basic questions more precisely in terms of equations, and show that they can be answered once the ceiling projection is known. We discuss interesting curves on cones that are not conic sections, including geodesic curves and curves cut by various cylinders.

Figure 15a shows two familiar curves on a cone. One is a circular cross section we call a *base*, whose unwrapped image is a circular arc, the dashed curve in Figure 15b. The other is an ellipse that unwraps to form a new plane curve, a *generalized ellipse* shown in Figure 15b.

We now replace the ellipse in Figure 15a with a general curve C lying on the cone and unwrap it onto a plane. In this plane, generators of the cone are mapped onto

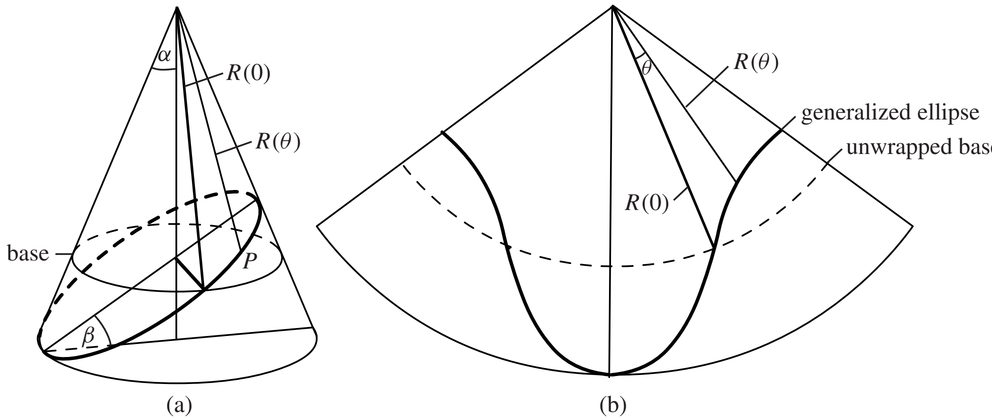


Figure 15. (a) An elliptical cross section; (b) its unwrapped image on a plane.

radial lines emanating from the cone's vertex. A point P on C is mapped onto a point in the plane with polar coordinates $(R(\theta), \theta)$, with the origin at the vertex of the cone, where $R(\theta)$ is the distance of P from the vertex of the cone, and θ is the polar angle in radians measured from a fixed radial line to that through the image of P. We can also regard θ as being measured along the surface of the cone from the fixed generator to the generator through P. Thus, $(R(\theta), \theta)$ can be thought of as conical coordinates on the cone itself.

The function $R(\theta)$ depends on C, and we formulate the following general problem:

Basic problem. *For a given curve C on the cone, obtain an explicit formula for $R(\theta)$. In particular, describe $R(\theta)$ when C is a conic section.*

The analysis on a cone differs from that on a cylinder, but again depends on arc length invariance, as indicated in the following special case.

10. UNWRAPPED BASE AND PRESERVATION OF ARCLENGTH. For any finite portion of the cone with a circular base, as shown in Figure 16a, the unwrapped image of the base is a circular arc with center at the cone's vertex and radius equal to the slant height s of that finite portion. In this simple case, the basic problem is easily solved because the radial distance $R(\theta)$ is constant, $R(\theta) = R(0) = s$.

Figure 16 also reveals a basic fact that plays a key role in solving the general problem. Let ρ denote the radius of the base in Figure 16a. When the base rolls through an angle of φ radians, the corresponding portion of the base of arclength $\rho\varphi$ unwraps onto a circular arc of radius s and central angle that we denote by θ (Figure 16b). Because the cone is a developable surface, distances are preserved when the cone is unrolled onto a plane, so we have

$$s\theta = \rho\varphi. \tag{14}$$

It is easy to see that the relation between θ and φ is independent of ρ and s. In Figure 16a, α is half the vertex angle of the cone, and ρ is related to s by the equation

$$\rho = s \sin\alpha. \tag{15}$$

Combining (14) and (15) we find a relation independent of ρ and s:

$$\theta = \varphi \sin\alpha. \tag{16}$$

The simple relation (16), with $0 < \alpha < \pi/2$, occurs repeatedly in analyzing the shape of any curve unwrapped from a cone. With $k = 1/\sin\alpha$, (16) can be written as

$$\varphi = k\theta. \tag{17}$$

Thus, the sine of half the vertex angle determines the relation between φ and θ.

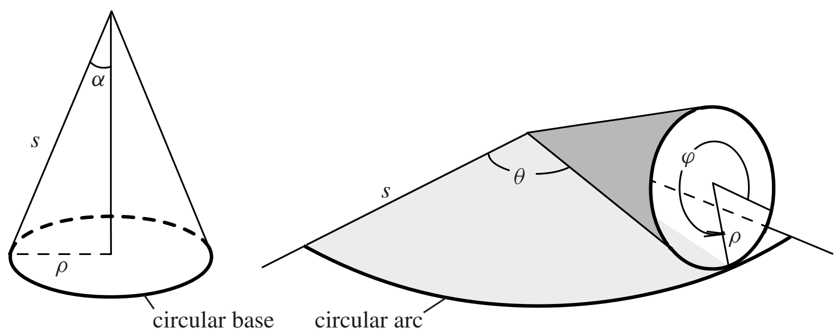


Figure 16. (a) Finite portion of a cone with slant height s; (b) unwrapping the surface of this finite portion.

When the cone is cut by a plane through a diameter of the base inclined at angle β, the conic section is an ellipse, parabola, or hyperbola, depending on β. Figure 15a shows an ellipse, which unwraps to form a *generalized ellipse* that oscillates about the image of the base as shown in Figure 15b. An explicit formula for $R(\theta)$, which depends on both β and the cone's vertex angle, is given in (23).

11. REFORMULATED PROBLEM IN TERMS OF CEILING PROJECTION.

In Theorem 1 we analyzed a general curve on a cylinder by projecting it onto an unwrapping plane parallel to the generators of the cylinder. To analyze a curve C lying on a cone, we project it upward onto the horizontal ceiling plane, a plane orthogonal to the axis of the cone and passing through its vertex V, as indicated in Figure 17.

Curve C projects onto a curve C_0 in the ceiling plane that we describe with a polar equation $r = r(\varphi)$, where $r(\varphi)$ is the radial distance measured from vertex V as origin. The ceiling projection C_0 is the profile of a vertical cylinder that intersects the cone along C. Figure 17 reveals that the two distances $R(\theta)$ and $r(\varphi)$ satisfy $r(\varphi) = R(\theta)\sin\alpha$, where α is half the vertex angle of the cone. This simple relation, together with (17), shows that the ceiling projection is the key that unlocks the basic problem.

Theorem 5. *Let C be a curve on the surface of a cone with vertex angle 2α. If the ceiling projection C_0 has polar equation $r = r(\varphi)$, then the unwrapped image of C has polar equation*

$$R(\theta) = kr(k\theta), \tag{18}$$

where $k = 1/\sin\alpha$. Conversely, if $R(\theta)$ is known, then (18) *determines $r(\varphi)$:*

$$r(\varphi) = R(\varphi/k)/k. \tag{19}$$

Proof. The relation $r(\varphi) = R(\theta)\sin\alpha$ becomes $r(\varphi) = R(\theta)/k$, which in view of (17) gives (18) and (19). ∎

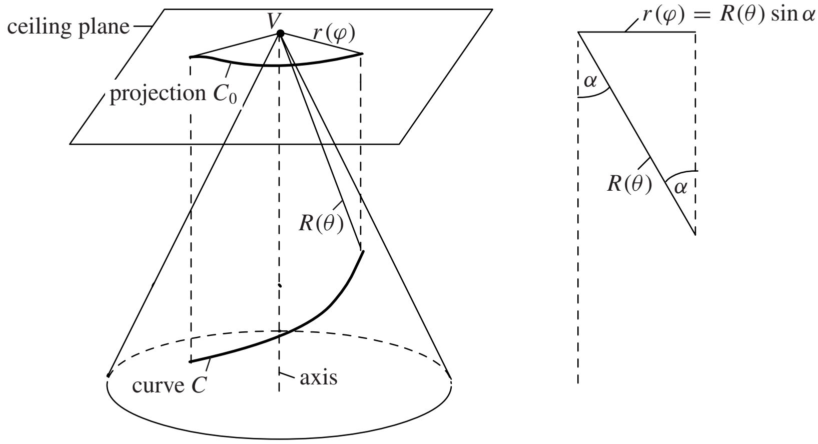


Figure 17. Curve C on the cone projects onto curve C_0 in the ceiling plane with polar equation $r = r(\varphi)$.

Note. Before discovering Theorem 5, we solved the basic problem for the special case of an unwrapped conic, using a lengthy "brute force" analysis of the solid geometry of the cone. To our surprise, the resulting formula for $R(\theta)$ in (23) resembled that of an ordinary conic. An analysis of this formula suggested introducing the ceiling projection, which applies not only to conic sections, but to any curve C lying on a cone.

12. CEILING PROJECTION AND UMBRELLA TRANSFORMATION. Equation (18) is the end result of three transformations: projecting C onto C_0, which produces $r(\varphi)$; stretching each radial distance $r(\varphi)$ by the factor k; and squeezing the polar angle φ by the factor $1/k$. The first two can be combined into one transformation given by

$$R(\theta) = kr(\varphi), \tag{20}$$

which is a scaled version of the ceiling projection. To visualize (20) geometrically, regard the cone as an "umbrella" that can be opened up flat onto the ceiling plane by rotating each generator vertically upward about vertex V, as shown in Figure 18. This rotation preserves radial distances from the vertex, but increases angles between generators on the cone. Two generators separated by an angle θ measured along the surface of the cone lie on two vertical planes through the axis making a dihedral angle φ, and during the rotation the umbrella transformation stretches the angle between them from θ to $\varphi = k\theta$.

13. CONE TO CONE. The analysis used to prove Theorem 5 also treats the more general case in which a curve C_1 on one right circular cone with vertex angle $2\alpha_1$ is unwrapped onto a curve C_2 on another right circular cone having the same vertex, but with vertex angle $2\alpha_2$. When the vertex angle $2\alpha_2$ is a straight angle, the second cone becomes an unwrapping plane. When we unwrap one cone onto another it is understood that we keep the cones tangent to each other along a common "rolling" generator. The next theorem, whose proof we leave to the reader, relates the ceiling projection functions of curves C_1 and C_2.

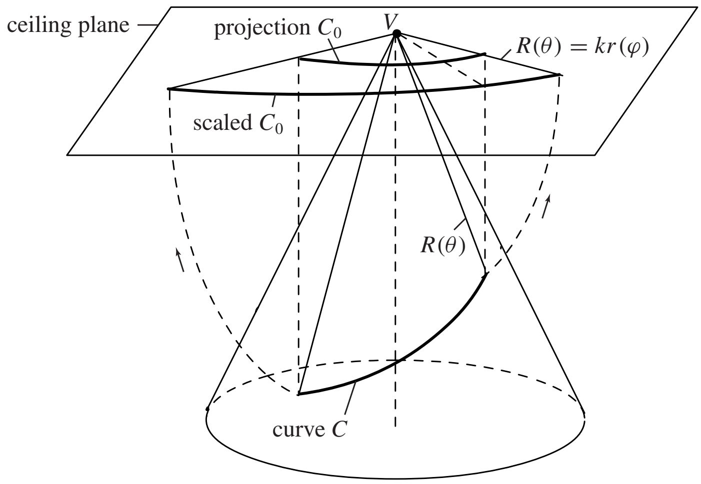


Figure 18. Umbrella transformation (20) maps curve C from the cone to a scaled version of projection C_0.

Theorem 6. *If the ceiling projection of curve C_1 has polar equation $r_1 = r_1(\varphi)$ and that of curve C_2 has polar equation $r_2 = r_2(\varphi)$, then the two functions satisfy*

$$r_2(\varphi) = \mu r_1(\mu\varphi), \tag{21}$$

where $\mu = \sin\alpha_2 / \sin\alpha_1$.

When $k_2 = 1$, cone 2 coincides with its ceiling plane, $\varphi = \theta$, $r_2 = R$, $r_1 = r$, and (21) turns into (18). Note also that $\mu < 1$ if cone 1 has a larger vertex angle than cone 2, and, vice versa, $\mu > 1$ if cone 2 has a larger vertex angle than cone 1.

14. UNWRAPPING A CONIC SECTION FROM A CONE ONTO A PLANE.

Now take C to be a conic section cut from a cone by a plane inclined at angle β with the ceiling plane, where $0 \le \beta < \pi/2$. The cutting plane intersects the axis of the cone at a point O that we use as the center of a circular base of radius ρ (Figure 19). As before, α is half the vertex angle of the cone, where $0 < \alpha < \pi/2$. In Figure 19, C is shown as an ellipse, but the analysis also applies to a parabola or hyperbola. The generator through point P on C intersects the base at point B whose polar coordinates in the plane of the base are (ρ, φ), where φ is measured from radius OA, with $A = (\rho, 0)$. The case $\beta = 0$ corresponds to unwrapping the base, which was treated earlier.

Figure 19 also shows the ceiling projection C_0 of conic C. The next theorem completely solves the unwrapping problem for a conic, and also includes a surprising result in part (a).

Theorem 7. *The ceiling projection of a conic section C is another conic C_0 with the following features:*

(a) *a focus at the vertex of the cone,*
(b) *a directrix at the line of intersection of the cutting plane and the ceiling plane,*
(c) *eccentricity $\lambda = \tan\alpha \tan\beta$,*
(d) *polar equation*

$$r(\varphi) = \frac{r(0)}{1 + \lambda \sin\varphi}. \tag{22}$$

Thus, if $k = 1/\sin\alpha$, the unwrapped image of C on a plane has polar equation

$$R(\theta) = \frac{R(0)}{1 + \lambda \sin(k\theta)}, \tag{23}$$

with $R(0) = kr(0)$.

Proof. Let L denote the line of intersection of the cutting plane and ceiling plane. For any point P on C let P_0 denote its projection on C_0. Let d be the distance from P_0 to L and r the distance from P_0 to V, as depicted in Figure 19.

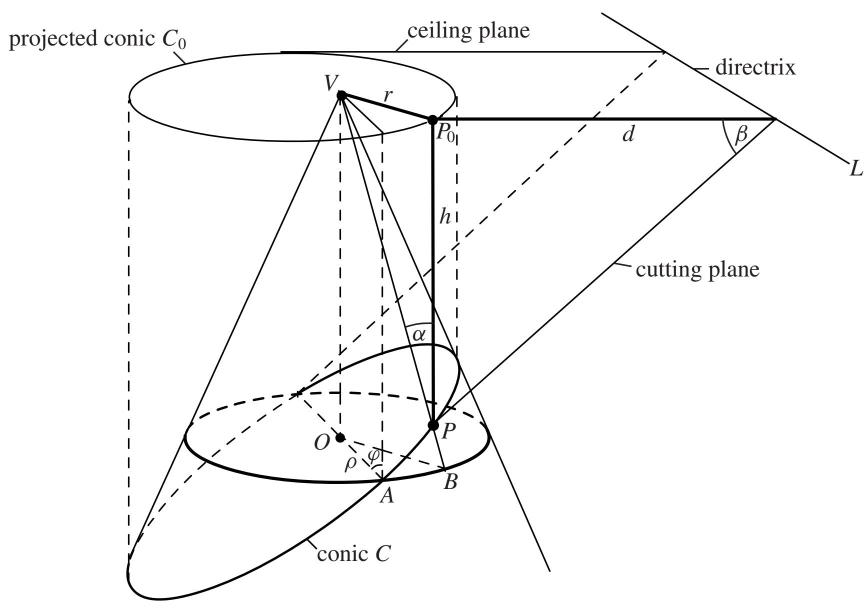


Figure 19. Diagram for proving parts (a), (b), and (c) of Theorem 7. (The ceiling plane intersects the cutting plane along the directrix L of the projected conic.)

We now show that the ratio r/d is $\tan\alpha \tan\beta$, a constant (independent of P_0) that we denote by λ. This will prove (a), (b), and (c). Write the ratio r/d as

$$\frac{r}{d} = \frac{r}{h}\frac{h}{d},$$

where h is the distance from P to P_0. From Figure 19 we see $r/h = \tan\alpha$ and $h/d = \tan\beta$, so the foregoing equation becomes $r/d = \tan\alpha \tan\beta$, as required. This ensures that C_0 is a conic with a focus at V and directrix L.

To derive (22), use the focus V as origin in the ceiling plane, and let $r(\varphi)$ denote the distance from V to the point on C_0 with polar coordinates $(r(\varphi), \varphi)$, where φ is measured from a line through V parallel to L, as shown in Figure 20. We have shown that C_0 is a conic with eccentricity λ, hence the focal definition of conic gives $r(\varphi) = \lambda d$. But $d = D - r(\varphi)\sin\varphi$, where D is the distance from the focus to the directrix, hence $r(\varphi) = \lambda(D - r(\varphi)\sin\varphi)$, which implies that $r(\varphi) = \lambda D/(1 + \lambda \sin\varphi)$. When $\varphi = 0$ we get $\lambda D = r(0)$, which proves (22), and (23) follows from Theorem 5. ∎

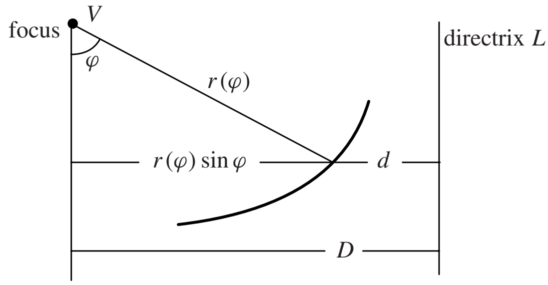


Figure 20. Diagram for deriving the polar equation of the projected conic C_0.

Note. The ratio $e = (\sin\beta)/(\cos\alpha)$ is known to be the eccentricity of the conic section C in Figure 19. It is easily verified that $e = \lambda = 1$ when $\alpha + \beta = \pi/2$, that $0 < e < 1$ and $0 < \lambda < 1$ when $\alpha + \beta < \pi/2$, and that $e > 1$ and $\lambda > 1$ when $\alpha + \beta > \pi/2$. Therefore conic C and its ceiling projection C_0 are of the same type: ellipse, parabola, or hyperbola. Although their eccentricities may differ, both are simultaneously less than 1, equal to 1, or larger than 1.

In Theorem 7, the relations between the parameters λ and k as well as the angles α and β imply the restrictions $\lambda \geq 0$ and $k \geq 1$. The inequality $\lambda \geq 0$ is not serious, because changing the sign of λ in (23) is equivalent to replacing θ with $-\theta$, which means the unwrapping occurs in the opposite direction. The restriction $k \geq 1$ is more serious because $k = 1/\sin\alpha$. However, (23) is meaningful for all real λ and k and gives a function $R(\theta)$, periodic in θ with period $2\pi/k$, that represents a well-defined curve, even if $k < 1$. This motivates the following notion of a generalized conic.

Definition. A plane curve described by a polar equation

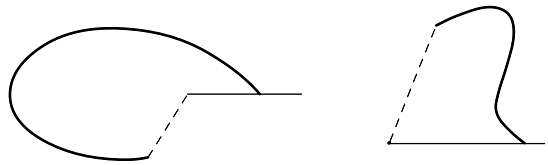
$$R(\theta) = \frac{R_0}{1 + \lambda\sin(k\theta)}, \tag{24}$$

where R_0 and λ are nonnegative constants and k is an arbitrary real constant, is called a *generalized conic*. The curve is called a generalized ellipse, parabola, or hyperbola, according as $\lambda < 1$, $\lambda = 1$, or $\lambda > 1$, respectively.

If $k = 1$, the cone is its ceiling plane, and (24) is the polar equation in this plane of a conic with eccentricity λ and with one focus at the origin. If $k > 1$, Theorem 7 tells us the curve in (24) is obtained by unwrapping a conic section of eccentricity λ from a cone with vertex at the origin and vertex angle 2α, where $\sin\alpha = 1/k$. If $k < 1$, the curve in (24) cannot be obtained by unwrapping a conic section from a cone onto a plane, but it can be realized as the ceiling projection of a curve C on a cone K' with vertex angle $2\alpha'$, where $\sin\alpha' = k$ and where C is obtained by wrapping a conic of eccentricity λ from a plane onto K'. In terms of equations, a conic with polar equation (22) is the unwrapped version of a curve C on a cone K' with vertex at a focus of (22) and with ceiling projection described by (24).

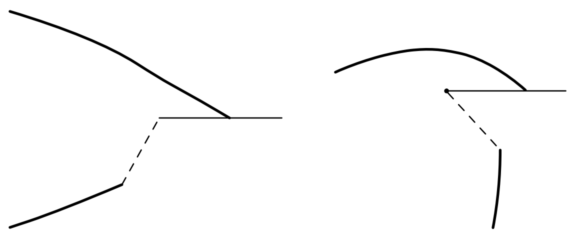
Incidentally, if a conic on one cone is wrapped onto another cone, the ceiling projection of the second cone is a generalized conic, as is seen from (21). Thus, wrapping from cone to cone is no more general than wrapping from cone to plane, and vice versa.

15. EXAMPLES OF GENERALIZED CONICS. Examples of these curves are shown in Figures 21 through 26. In Figures 21 to 23, the angle θ runs through one



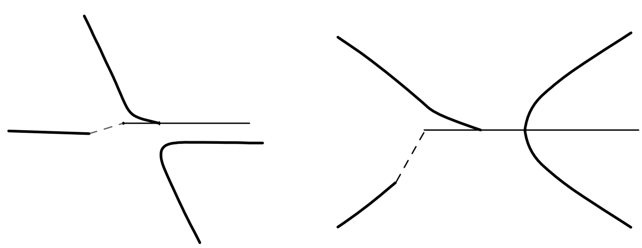
$k = 1.5, \lambda = 0.6$ $\qquad$ $k = 5.5, \lambda = 0.22$

Figure 21. Generalized ellipses, one period only. (For more periods, see Figure 24.)



$k = 1.5, \lambda = 1$ $\qquad$ $k = 1.15, \lambda = 1$

Figure 22. Generalized parabolas, one period.



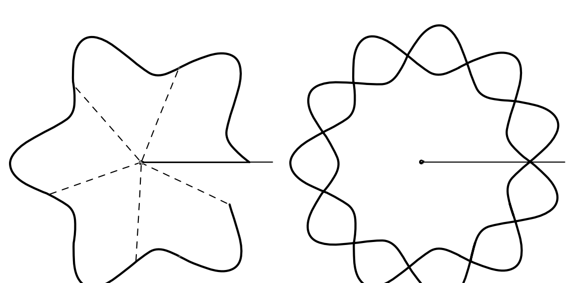
$k = 1.5, \lambda = 1.6$ $\qquad$ $k = 1.83, \lambda = 1.8$

Figure 23. Generalized hyperbolas, one period. Both nappes are cut and unwrapped onto a plane.

period interval of length $2\pi/k$, and in Figure 24 through more than one period interval. In Figures 21 to 24, each example has $k > 1$ and can be obtained by unwrapping a conic section from a cone onto a plane.

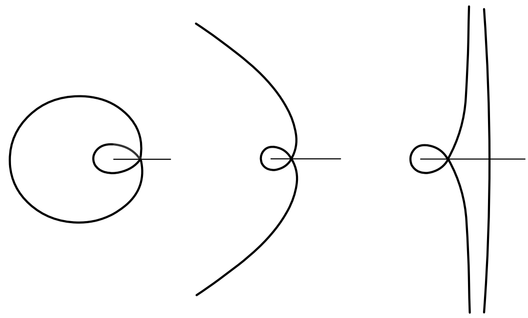
In Figures 25 and 26, each example has $k < 1$ (namely, $k = 1/2$) and cannot be obtained by unwrapping a conic from a cone.

You can construct more examples for yourself by visiting the following interactive web site: `http://www.its.caltech.edu/~mamikon/genconic.html`. A curve with $k > 1$



five periods $\qquad$ eleven periods

Figure 24. Second generalized ellipse from Figure 21 ($k = 5.5, \lambda = 0.22$), with more than one period.



(a) generalized ellipse, λ = 0.8 (b) generalized parabola, λ = 1 (c) generalized hyperbola, λ = 1.4

Figure 25. Generalized conics with $k = 0.5$ in (24). (They cannot be obtained by unwrapping a conic from a cone, but each is the ceiling projection of a conic wrapped from a plane onto a cone.)

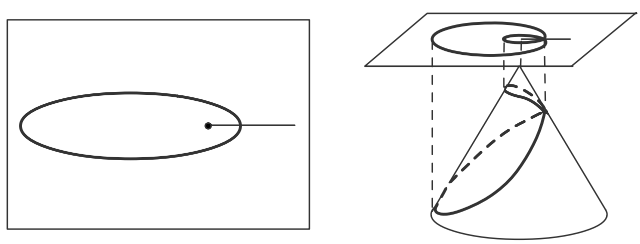


Figure 26. Generalized ellipse in Figure 25a as the ceiling projection of a curve obtained by wrapping an ellipse onto a cone with vertex angle $\pi/3$ to form a curve with two nonplanar loops.

can be used as a template for cutting and rolling a piece of paper into a cone having a conic as a plane cross section. For example, rolling a template made from any curve in Figure 21 can produce a right circular cone with the rolled curve becoming a planar elliptical cross section.

Note. If the ceiling projection C_0 of a curve C on a cone is a conic, C itself need not be a conic. In fact, it is easy to verify that C is a conic if and only if a focus of C_0 is at the vertex of the cone.

16. LIMITING CASES. From a given cone one can obtain all possible ellipses and parabolas as conic sections. This is not true of hyperbolas: only those arise whose asymptotes intersect at an angle smaller than the vertex angle of the cone. By contrast, all possible conics can be obtained as limiting cases of (23), as will be shown presently.

First we show that sinusoidal curves unwrapped from circular cylinders are limiting cases of generalized conics. In the plane of the unwrapped cone (Figure 15b), the difference $y = R(0) - R(\theta)$ is the radial distance from the image of the circular base to the generalized conic. From (24) we have $R_0 = R(0)$, and we obtain

$$y = R(0)\frac{\lambda \sin(k\theta)}{1 + \lambda \sin(k\theta)}. \qquad (25)$$

Keep the radius ρ of the circular base fixed and keep the angle of inclination β fixed, but let $\alpha \to 0$ so that the vertex of the cone recedes to infinity. Then the cone becomes a cylinder of radius ρ (which can be regarded as the limiting case of a cone). What happens to the right-hand side of (25) as $\alpha \to 0$? For small α we can approximate

$\tan\alpha$ by $\sin\alpha$, so $\lambda = \tan\alpha \tan\beta$ can be approximated by $\sin\alpha \tan\beta$. The denominator of (25) is very close to 1, so the right side of (25) has the approximate value

$$R(0) \sin\alpha \tan\beta \sin\varphi,$$

where $\varphi = k\theta$. But, in view of (15), $R(0)\sin\alpha = \rho$, hence (25) is nearly the same (for small α) as the limiting relation

$$y = \rho \tan\beta \sin\varphi = \rho \tan\beta \sin(x/\rho),$$

where x is the length of arc subtended by an angle φ on a circle of radius ρ. This is a Cartesian equation of a sinusoidal curve cut from a circular cylinder of radius ρ by a plane inclined at angle β. In other words, if the circular base of the cone is kept fixed while the vertex recedes to infinity, the cone becomes a cylinder, and the generalized ellipse unwrapped from the cut cylinder becomes a sinusoidal curve, as noted in section 1.

The other limiting case is when $\alpha \to \pi/2$ and the cone flattens onto the ceiling plane. In this case we keep $R(0)$ and λ fixed, meaning that $\tan\beta = \lambda/\tan\alpha$. Then $\tan\alpha \to \infty$, $\beta \to 0$, and $k = 1/\sin\alpha \to 1$, so the limiting value of polar equation (24) is

$$R(\theta) = \frac{R(0)}{1 + \lambda \sin\theta}. \tag{26}$$

This describes an ordinary conic section of eccentricity λ, in which $R(\theta)$ is the distance from a focus to the point $(R(\theta), \theta)$ on the conic, as illustrated in Figure 20. Geometrically, as $\alpha \to \pi/2$ and the cone flattens onto its ceiling plane, the conic section of eccentricity λ turns into the conic described by (26), which is the ceiling conic (22). Stated differently, as the cone flattens onto a plane, the limiting case of the conic section coincides with the ceiling conic having its focus at the vertex, and it also coincides with the limiting case of the generalized conic. All three types of conics—ellipse, parabola, and hyperbola—with all possible values of the eccentricity, can occur in this limiting case.

17. OTHER CURVES ON A CONE. Conic sections are widely studied, but there are other interesting curves on a cone that we can analyze using our transformations. In each of the next two examples, the curve is determined by specifying its ceiling projection to be a spiral, and we find that the unwrapped curve is a spiral of the same type.

Example 12 (Archimedean spiral C_0). Here $r(\varphi) = c\varphi$ for some constant $c > 0$. In the ceiling plane, spiral C_0 intersects a given radial line at equidistant points, with distance $2\pi c$ between consecutive intersections. Curve C spirals around the cone as shown in Figure 27. From (18) we find that

$$R(\theta) = ck^2\theta,$$

so the unwrapped curve is another Archimedean spiral with ck^2 replacing c.

Example 13 (Logarithmic spiral C_0). Now $r(\varphi) = Ae^{c\varphi}$ for positive constants A and c. In the ceiling plane, the tangent line to the spiral at each point makes a constant angle δ with the radial line to that point, where $\cot\delta = c$. From (18) we obtain

$$R(\theta) = kAe^{ck\theta},$$

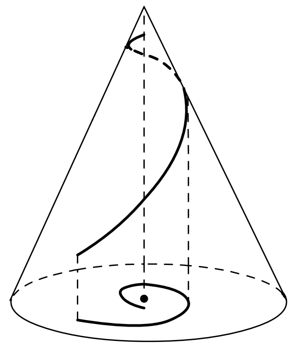


Figure 27. A conical spiral with an Archimedean spiral as ceiling projection and as unwrapped curve.

hence the unwrapped curve is another logarithmic spiral with new constants. Its tangent line makes a constant angle ψ with the radial line, where $\cot\psi = ck$.

Example 14 (Geodesic on a cone). Because distances are preserved when a cone is unwrapped, the image of a geodesic arc on a cone (the shortest path joining two points on the surface) is a line segment in the plane of the unwrapped cone. To construct a geodesic curve on a cone, start with a straight line and wrap it onto the cone. Figure 28a shows a line L and a point V not on L that we take as the vertex of an unwrapped cone. The entire line has polar equation

$$R(\theta) = \frac{d}{\cos\theta},$$

where d is the shortest distance from V to the line, and θ (measured as indicated) varies from $-\pi/2$ to $\pi/2$. The plane determined by the line and point V can be rolled into many right circular cones with V as a common vertex but with different vertex angles, and the line is mapped onto a geodesic curve on each such cone. If the cone has vertex angle 2α, the ceiling projection of this geodesic has polar equation

$$r(\varphi) = \frac{d/k}{\cos(\varphi/k)},$$

where $k = 1/\sin\alpha$ and $-k\pi/2 < \varphi < k\pi/2$. Figures 28b and c show one example of a geodesic and its ceiling projection. Figure 29 shows more ceiling projections.

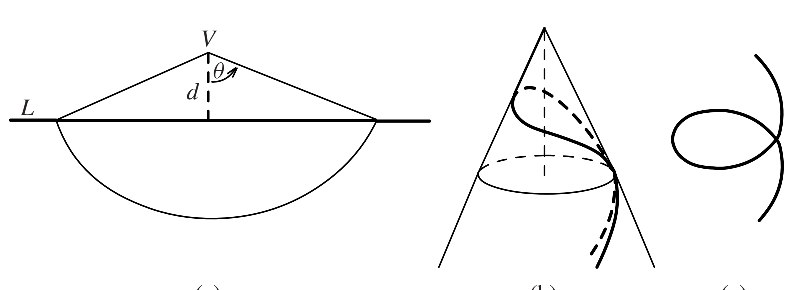


Figure 28. Line segment (a) wrapped onto a geodesic (b) on a cone, with ceiling projection (c).

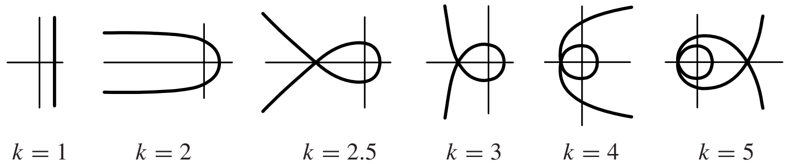


Figure 29. Ceiling projections of one line wrapped onto cones with different vertex angles ($\sin\alpha = 1/k$).

18. VERTICAL WALL PROJECTION. Analyzing the unwrapped version of a curve on the surface of a cone is equivalent, by Theorem 5, to finding its ceiling projection. We turn now to examples in which the curve C is the intersection of the cone with a horizontal cutting cylinder whose generators are parallel to the ceiling plane. The projection of C on a vertical plane perpendicular to the generators is a profile of the cylinder.

We choose such a plane through the axis of the cone and call it the *wall plane.* It intersects the ceiling plane along a line that we designate as the t-axis, with its origin at vertex V, as shown in Figure 30. The axis of the cone is designated as the z-axis, but with its positive direction pointing down as indicated in Figure 30. If the cone is flipped over and the ceiling plane becomes a horizontal "floor" plane, the coordinate axes of the wall plane will be in traditional position, with the positive t-axis pointing to the right, and the positive z-axis pointing up.

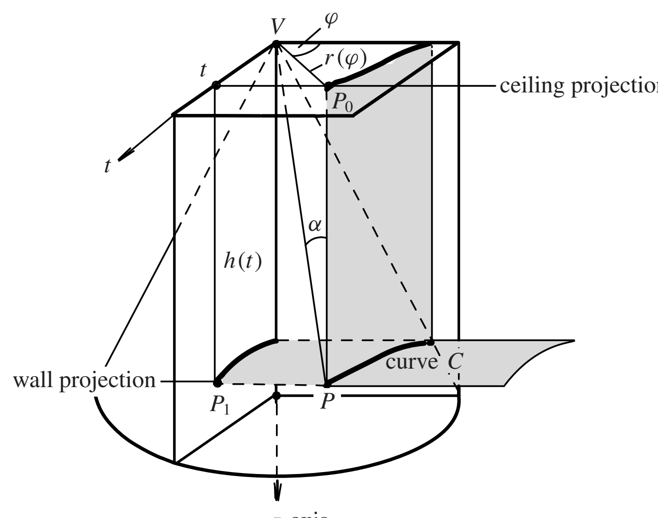


Figure 30. Relating the wall projection and ceiling projection of a curve on a right circular cone.

The cutting cylinder intersects the wall plane along a profile curve with implicit equation of the form $p(t, z) = 0$. We call this curve the *wall projection* of C. A general point P on C has as its ceiling projection the point P_0 with polar coordinates (r, φ), where φ is measured as indicated in Figure 30 and $r = r(\varphi)$ is the ceiling function. Point P also has wall projection P_1 with coordinates (t, z) related by $p(t, z) = 0$. The next theorem, which follows at once from Figure 30, relates the coordinates (t, z) of P_1 to the polar coordinates (r, φ) of P_0.

Theorem 8. *On a right circular cone with vertex angle 2α, let $c = \tan\alpha$, and let C be a curve with ceiling projection function $r = r(\varphi)$ and wall profile $p(t, z) = 0$. Then the coordinates are related as follows:*

$$t = r\sin\varphi, \qquad z = \frac{r}{c}. \tag{27}$$

Consequently,

$$\varphi = \arcsin\left(\frac{t}{r}\right), \qquad r = cz, \tag{28}$$

and

$$p(r\sin\varphi, r/c) = 0.$$

In particular, if the wall profile gives z explicitly as a function of t, say $z = h(t)$, then

$$r(\varphi) = ch(r(\varphi)\sin\varphi) \tag{29}$$

and

$$ch(t) = r\left(\arcsin\left(\frac{t}{ch(t)}\right)\right). \tag{30}$$

The following examples specify the wall projection and determine $r(\varphi)$:

Example 15 (Linear wall projection: $h(t) = at + b$). The cutting cylinder in this case is a plane, and (29) becomes

$$r(\varphi) = c(ar(\varphi)\sin\varphi + b),$$

which can be solved for $r(\varphi)$ to yield

$$r(\varphi) = \frac{bc}{1 - ac\sin\varphi}.$$

As expected, this is the polar equation of a conic section.

Example 16 (Circular cutting cylinder of radius 1). Cut the cone with a circular drill of radius 1, perpendicular to the axis of the cone, whose center in the tz-plane is at the point (w, d). Then the wall projection satisfies the implicit equation

$$(t - w)^2 + (z - d)^2 = 1.$$

From (27) we find $(r(\varphi)\sin\varphi - w)^2 + (r(\varphi)/c - d)^2 = 1$, which is quadratic in $r(\varphi)$.

If $w = 0$, the axis of the cutting cylinder passes through the axis of the cone and the quadratic equation simplifies to

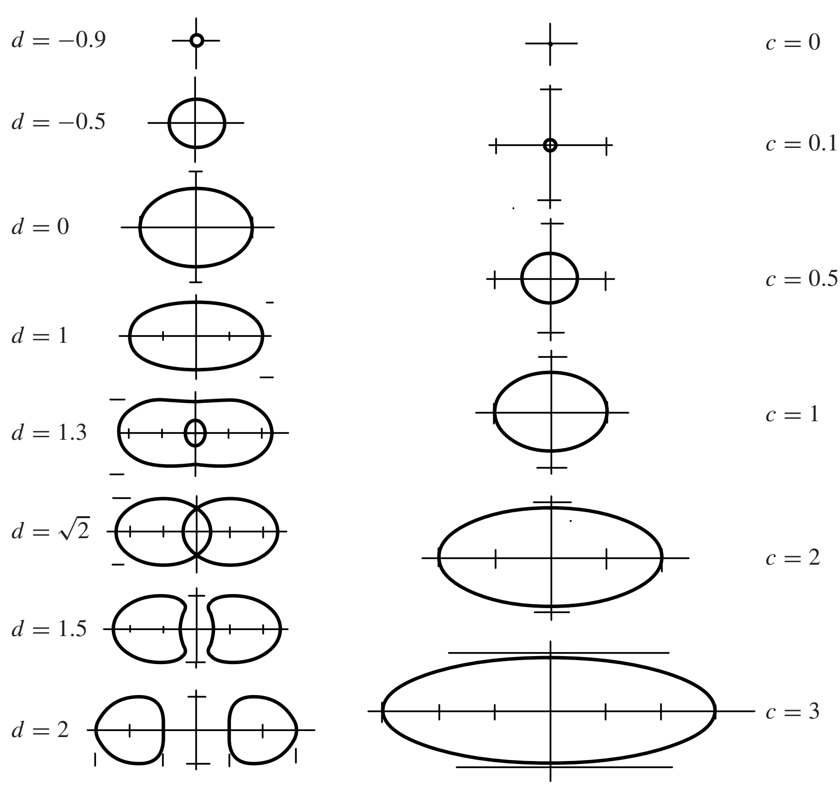
$$(r(\varphi)\sin\varphi)^2 + \left(\frac{r(\varphi)}{c} - d\right)^2 = 1. \tag{31}$$

A graphing calculator can draw the graphs of (31), revealing the ceiling projection of the hole for various values of c and d. Figure 31a shows snapshots on one nappe only for $c = 1$ and increasing d, while Figure 31b shows snapshots for $d = 0$ and increasing c.

Powerful 3-D modeling programs can be used to render the qualitative shape of the curve of intersection of a cone and a cutting cylinder. But exact equations like those derived here provide a deeper understanding and are also useful when graphing projection functions such as those in Figure 31 with simple 2-D programs. These graphs are not specified by 3-D modeling programs that, for example, do not reveal whether a projected oval curve is an ellipse or a curve of higher degree. Knowing that a curve is an ellipse can have profound implications. For example, Kepler's landmark discovery that planetary orbits are elliptical implies Newton's inverse-square law of gravitation.

The exact equations allow us to plot curves easily and animate them on a computer screen using simple 2-D graphics programs instead of 3-D programs. Most illustrations in this paper were prepared in this manner.

In Figure 31a the vertex angle of the cone is $\pi/2$. For small d the projection is a centrally symmetric oval curve, which gradually changes its size and shape as d increases. At some stage the center of the oval is pierced by a hole that increases in size until $d = \sqrt{2}$, when the ceiling projection consists of two overlapping confocal ellipses. As d increases further, the ceiling projection splits into two disconnected symmetric pieces that move further apart. This is consistent with our intuitive idea of how the hole changes as a drill of constant radius passes through the axis of the cone



(a) $\alpha = \pi/4$ (b) $c = \tan\alpha$

Figure 31. Snapshots of ceiling projection of a horizontal hole of radius 1 drilled through the axis. (In (a) the cone is fixed and the coordinate d varies; in (b) the drill is fixed and the vertex angle changes.)

but continues to move away from the vertex. In Figure 31b, the axis of the drill passes through the vertex of the cone and the snapshots show how the ceiling projection of the hole varies as the vertex angle of the cone increases. We were surprised to learn that all the projected curves in Figure 31b are ellipses! This is easily verified by writing (31) in rectangular coordinates.

19. TILTED WALL PROJECTION. In the foregoing discussion, the cutting cylinder had its generators parallel to the ceiling plane. In descriptive geometry, and in applications to sheet metal work, the cutting cylinder is not always parallel to the ceiling plane but may be tilted at an angle β. By rotating the wall plane about the t-axis we can relate the profile of C on the tilted plane with the wall and ceiling projections. The relation is provided by the following theorem, whose proof is omitted:

Theorem 9. *A tilted cylinder whose generators make an angle β with the ceiling plane and that has tilted profile function $z' = q(t)$ intersects a cone of vertex angle 2α along a curve whose wall profile $z = h(t)$ is related to $q(t)$ as follows:*

$$q(t) = h(t)\cos\beta + \sqrt{c^2h^2(t) - t^2}\,\sin\beta, \tag{32}$$

where $c = \tan\alpha$.

This expresses the tilted profile $q(t)$ directly in terms of the wall profile $h(t)$ and the angles α and β. Also, if $q(t)$ is given, we can use (32) to find $h(t)$ by solving a quadratic equation. For example, if $q(t) = at + b$, the slanted cutting cylinder is a plane. If we put $z = h(t)$, then (32) becomes quadratic in z and t, and the horizontal profile is a conic, as expected.

20. ARCLENGTH AND AREA. When a curve of length L on a cylinder or cone is unwrapped onto another curve, the arclength L remains unchanged because distances are preserved. For example, the generalized ellipse in Figure 15b has the same length as the ellipse in Figure 15a, even though there is no simple formula for calculating these arclengths. In general, *any unwrapped curve has the same length as its wrapped version on the cone*. From Figure 1 we see that *an ellipse has the same arclength as an unwrapped sine curve*, without the need to calculate the elliptic integrals that produce numerical values.

Unwrapping also preserves areas. In Figure 16, a portion of the circular cone unwraps onto a circular sector with central angle θ (Figure 16b) with area $s^2\theta/2$. In terms of the parameters of the cone this area is equal to $\rho s\varphi/2$ by (14). In particular, when $\varphi = 2\pi$ each of these areas is equal to $\pi\rho s$.

More generally, we can ask for the sectorial area $A(\theta_1, \theta_2)$ of the region bounded by the unwrapped image of any curve C on the cone and two rays $\theta = \theta_1$ and $\theta = \theta_2$ emanating from the origin. When $\theta_1 < \theta_2$ this area is given by the following integral:

$$A(\theta_1, \theta_2) = \frac{1}{2}\int_{\theta_1}^{\theta_2} R^2(\theta)\,d\theta, \tag{33}$$

with $R(\theta)$ determined by (18). Because areas are preserved when unwrapping a cone, this integral has the same value as the lateral surface area of the portion of the cone between the vertex and the original curve C on the cone. The change of variable $\varphi = k\theta$ transforms the integral in (33) to

$$A(\theta_1, \theta_2) = \frac{1}{2k}\int_{k\theta_1}^{k\theta_2} R^2(\varphi/k)\,d\varphi.$$

In view of (19) we can write this as

$$A(\theta_1, \theta_2) = k\left(\frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)\, d\varphi\right), \tag{34}$$

where $r(\varphi)$ is the ceiling projection function for C_0 and $\varphi_i = k\theta_i$. The factor multiplying k in (34) is the sectorial area in the ceiling plane bounded by C_0 and two rays $\varphi = \varphi_1$ and $\varphi = \varphi_2$. Equation (34) can be rephrased in the following form:

Theorem 10. *On a cone of vertex angle 2α, the lateral surface area of the portion of the cone between the vertex and an arc of the original curve C on the cone is k times the area of the corresponding ceiling projection, where $k = 1/\sin\alpha$.*

This is to be expected heuristically, because a thin nearly flat triangle of area T formed by two nearby generators on the lateral surface of the cone, with one vertex at the vertex of the cone, projects onto the ceiling plane onto a triangle with area $T_0 = T\sin\alpha$, hence $T = kT_0$.

Although Theorem 10 refers to an unwrapped sectorial region, it implies a more general result for regions lying between two curves C_1 and C_2 (with corresponding unwrapping functions R_1 and R_2) and two given rays $\theta = \theta_1$ and $\theta = \theta_2$. The integral

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} |R_2^2(\theta) - R_1^2(\theta)|\, d\theta$$

gives the area of both the unwrapped region and the corresponding region on the cone. Each of these areas is k times the area of the corresponding ceiling projection. Again, this is to be expected heuristically, for any nearly flat elemental region of area T on the cone projects onto the ceiling plane as a region of area $T_0 = T\sin\alpha$, so $T = kT_0$. The next example gives a surprising consequence of this result.

Example 17 (Surface area pierced from a cone by a vertical cylinder). The foregoing geometric argument supporting Theorem 10 gives an unexpected result. Any region on the surface of the cone pierced by a vertical cylinder (with generators parallel to the axis of the cone) has area k times that of its ceiling projection, even if the cylinder is not circular. In other words, *the surface area removed from a cone by a vertical cylinder of constant horizontal cross-sectional area A is equal to kA, regardless of the shape or position of the cylinder.*

We conclude with an interesting observation concerning the elliptical cross section cut by a plane inclined at angle β (Figure 15a). Let E denote the area of the elliptical disk, and let S denote the lateral surface area of the finite portion of the cone with vertex angle 2α cut off by this disk. On the one hand, the ceiling projection of the ellipse has area $E\cos\beta$, and on the other hand, it is $S\sin\alpha$. This simple result for the ellipse must surely be known, but we could not find it in the literature. It deserves to be better known, so we state it here as a theorem.

Theorem 11. *The area E of an elliptic cross-sectional disk and the lateral surface area S of the finite portion of the cone cut off by the plane of the disk satisfy $S\sin\alpha = E\cos\beta$.*

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A Simple Proof for a Well-Known Fact

Inspired by "Tom Apostol's beautiful geometrical proof" for $\sqrt{2}$ being irrational (see [**1**]), we give a simple proof for a well-known fact:

Theorem. *For a positive integer N that is not a perfect square, $\sqrt{N}$ is irrational.*

Proof. Suppose $\sqrt{N} = a/b$, where a and b are positive integers and b is minimal. Then $\gcd(a, b) = 1$ and $a > b > 1$. Clearly b does not divide a, so there is some integer q such that $0 < a - qb < b$. The key observation is

$$\sqrt{N} = \frac{a}{b} = \frac{Nb}{a}$$

which implies that

$$\sqrt{N} = \frac{Nb - qa}{a - qb},$$

using the fact that if $\alpha/\beta = \gamma/\delta$ and $r\beta + s\delta \neq 0$ then

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} = \frac{r\alpha + s\gamma}{r\beta + s\delta}.$$

But $a - qb < b$, so this contradicts the minimality of b.

Alternatively, we can find integers r and s such that $ra + sb = 1$. Then

$$\sqrt{N} = \frac{a}{b} = \frac{Nb}{a} = \frac{rNb + sa}{ra + sb} = rNb + sa \in \mathbb{Z},$$

contradicting the fact that N is not a perfect square. ■

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