Subtangents—An Aid to Visual Calculus

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1. MAMIKON'S THEOREM. Areas and volumes of many classical regions can be determined by an intuitive geometric method that does not require integral calculus or even equations for the boundaries. The method and its many applications was first announced by Mamikon A. Mnatsakanian in [**1**], but this paper seems to have escaped notice, probably because it appeared in Russian in an Armenian journal with limited circulation. A recent exposition is given in [**2**], expanding on special cases discussed in [**3**] and [**4**]. The work is based on Mamikon's theorem illustrated in Figure 1. The lower curve on the left of Figure 1 is a more or less arbitrary smooth plane curve. A set of tangent segments to the lower curve defines a region that is bounded by the lower curve and an upper curve traced out by the other extremity of the tangent segments. This set is called a tangent sweep. It can be visualized dynamically as the region swept out by tangent segments moving along the lower curve. When each point of tangency is brought to a common point as shown on the right of Figure 1, the set of translated segments is called the tangent cluster. Mamikon's theorem states that the area of the tangent cluster is equal to that of its tangent sweep.

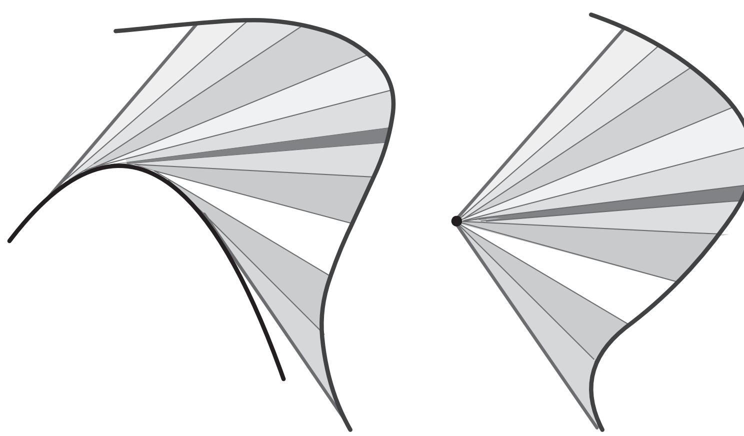

Figure 1. The tangent sweep of a curve and its tangent cluster have equal areas.

The theorem also holds for space curves and can be proved using differential geometry (see Section 5), but you can easily convince yourself that it is true by considering corresponding equal tiny triangles translated from the tangent sweep to the tangent cluster as suggested in Figure 1. (In three-space, the tangent cluster lies on a conical surface.)

When the tangent segments have constant length, the tangent cluster is a circular sector whose radius is that constant length. The example in Figure 2 equates the area of an annular ring (the tangent sweep of the inner circle) with that of a circular disk (the tangent cluster). It reveals the striking result that the area of the ring depends only on the length of the tangent to the inner circle and not on its radius. Incidentally, this example provided the original impetus that led Mamikon (as an undergraduate at the University of Yerevan in 1959) to develop his general theorem.

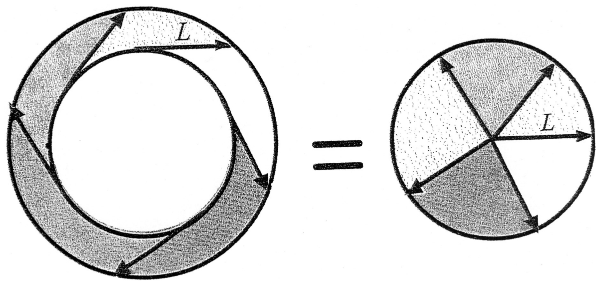

Figure 2. The area of an annular ring is equal to that of a circular disk.

The use of Mamikon's theorem to determine the area of more general oval rings is described in [**3**], and in [**2**] it is used to find the area of a parabolic segment, the area of a region under a general exponential curve, the area of the region under one arch of a cycloid, and the area of a "bicyclix," the region between two curves traced out by the front and rear wheels of a moving bicycle (the tractrix being a special case). All these applications also appear in [**1**]. The treatment of the parabolic segment in [**2**] and the exponential in [**4**] uses geometric properties of subtangents to these curves. (Subtangents are defined in Section 3.) This paper explores geometric properties of subtangents in greater detail. One of these properties reveals an unexpected connection between the tractrix and exponential curves (Section 4). Applications to volumes of solids appear in [**1**].

2. DRAWING TANGENT LINES TO EXPONENTIAL CURVES. The tangent line at a point $(x, f(x))$ on the graph of a function f is the line through that point with slope $f'(x)$. The simplest way to draw this line in practice, whether it's done by hand or on a computer, is to join the point $(x, f(x))$ with another point known to be on the tangent line. Sometimes one can find such a point without explicitly calculating the slope $f'(x)$.

We illustrate with three exponential curves. In Figure 3a, the line joining the point $(x-1, 0)$ with (x, e^x) is tangent to the graph of $f(x) = e^x$ because it has the required slope, $f'(x) = e^x$. In Figure 3b, the tangent joins $(x - \frac{1}{2}, 0)$ with (x, e^{2x}), and in Figure 3c the tangent joins $(x+1, 0)$ with (x, e^{-x}).

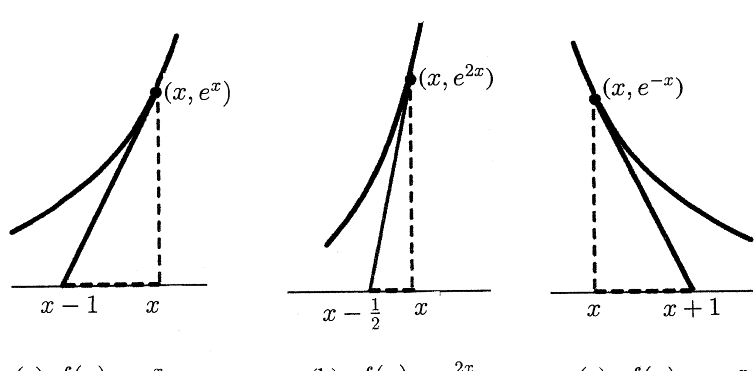

Figure 3. The simplest way to draw tangent lines to exponential curves.

3. SUBTANGENTS USED TO DRAW TANGENT LINES. On a general curve $y = f(x)$, the magic point on the x-axis that also lies on the tangent line through $(x, f(x))$ is given by $(x - s(x), 0)$, where $s(x)$ is the *subtangent* defined by

$$s(x) = \frac{f(x)}{f'(x)} \tag{1}$$

at each point x where the derivative $f'(x)$ is nonzero. In Figure 4, $s(x)$represents the base of a right triangle of altitude $f(x)$ and hypotenuse of slope $f'(x)$.

From (1) we find $f'(x) = f(x)/s(x)$, so if $s(x)$ is known, this gives a simple geometric procedure for finding the tangent line to an arbitrary point on the graph of f. As in Figure 4, drop a perpendicular from $(x, f(x))$ to the point $(x, 0)$ on the x-axis. Move to the point $(x - s(x), 0)$ on the x-axis, and join it to $(x, f(x))$ to get the required tangent line.

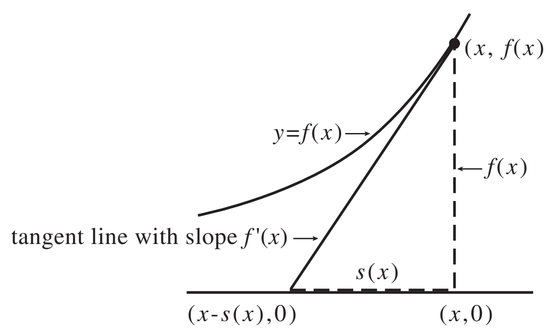

Figure 4. Geometric meaning of the subtangent. The tangent line at $(x, f(x))$ passes through the point $(x - s(x), 0)$ on the x-axis.

The method of construction is especially useful when $s(x)$ has a simple form.

Example 1 (Constant subtangents). Exponential curves were first introduced in 1684 when Leibniz posed the problem of finding all curves with constant subtangents. The solutions are the exponential curves. Specifically, given a nonzero constant b we have $f(x) = Ke^{x/b}$ for some constant $K \neq 0$ if and only if $s(x) = b$. Examples with $b = 1$, $1/2$, and -1 are shown in Figure 3. Incidentally, multiplying a function f by a nonzero constant does not alter its subtangent because f' is multiplied by the same factor, which cancels in (1).

The constant subtangent property of exponential curves was used in [**4**] together with Mamikon's theorem to show visually that the area of the region between the graph of $y = e^{x/b}$ and an arbitrary interval $(-\infty, x]$ is $be^{x/b}$, the same result one would get by integration.

Example 2 (Linear subtangents). Power functions have linear subtangents. In fact, for a nonzero constant b we have $f(x) = Kx^{1/b}$ for a constant $K \neq 0$ if and only if $s(x) = bx$. In particular, the parabola $f(x) = x^2$ has subtangent $s(x) = x/2$, and the hyperbola $f(x) = 1/x$ has subtangent $s(x) = -x$. Figure 5a shows how tangent lines to the parabola $f(x) = x^2$ can be easily constructed by joining $(x/2, 0)$ to (x, x^2). Figure 5b illustrates the construction for the hyperbola $f(x) = 1/x$. Here $x - s(x) = 2x$, so the tangent line passes through $(2x, 0)$ and $(x, 1/x)$.

For the cubic curve $f(x) = x^3$ we have $s(x) = x/3$, so we join $(2x/3, 0)$ with (x, x^3) to get the tangent line at (x, x^3), as illustrated in Figure 6a. For the more general

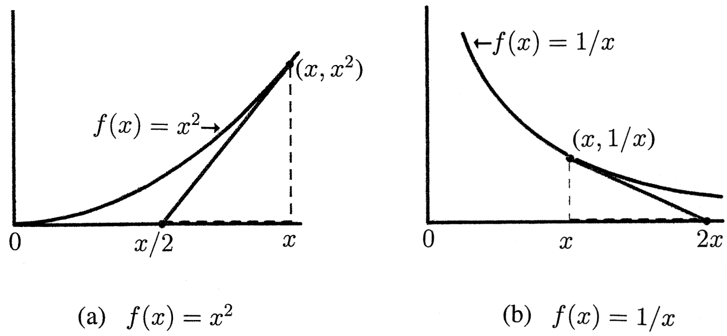

Figure 5. Simple geometric construction of tangents to (a) $f(x) = x^2$ and (b) $f(x) = 1/x$.

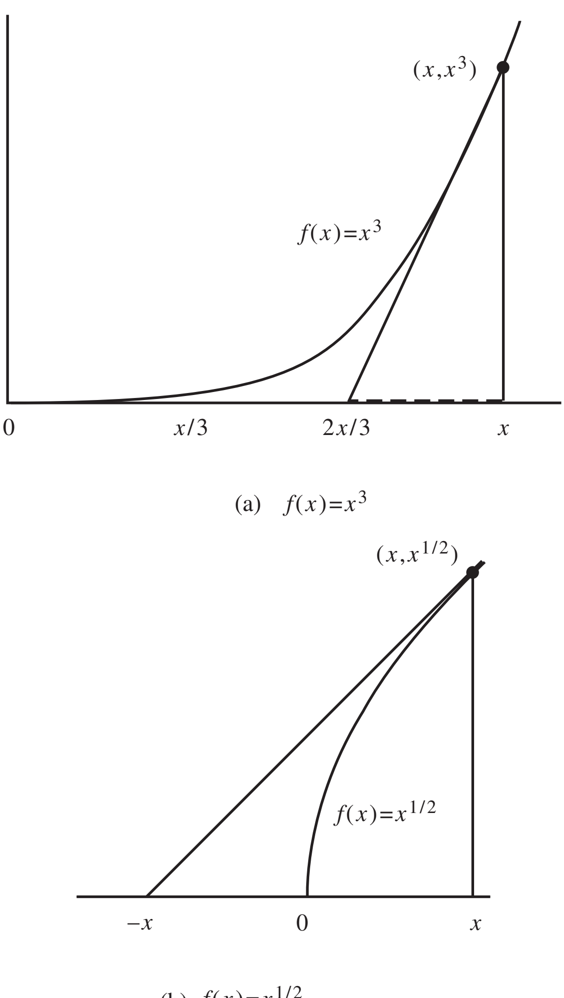

Figure 6. Simple geometric construction of tangents to (a) $f(x) = x^3$ and (b) $f(x) = x^{1/2}$.

power function $f(x) = x^r$ we join $(x - x/r, 0)$ with (x, x^r) to get the tangent line at (x, x^r). Figure 6b furnishes an example with $r = 1/2$.

The idea can be adapted to draw the tangent line to logarithmic curves. For example, because the logarithm $f(x) = \log x$ is the inverse of the exponential in Figure 3a (whose subtangent is always 1), the tangent line to the graph of the logarithm at $(x, \log x)$ passes through the point $(0, -1 + \log x)$ on the y-axis, as shown in Figure 7. As noted earlier, in [**4**] we used Mamikon's theorem to show that the area of the region between the graph of the exponential curve $y = e^x$ and the interval $(-\infty, t]$ is e^t. Therefore the area of the region in Figure 7 lying to the left of the logarithmic curve and to the right of the interval $(-\infty, \log x)$ on the y-axis is $e^{\log x} = x$. From this fact the reader can verify geometrically that the area of the region under the graph of the logarithmic curve in Figure 7 and above the interval $[1, x]$ is $x \log x - x + 1$, the same result obtained by integration.

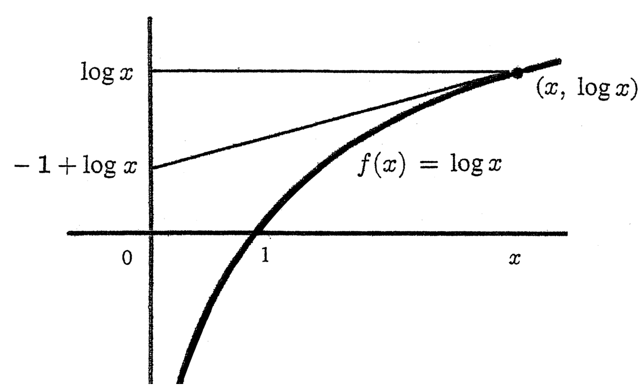

Figure 7. The tangent to $f(x) = \log x$ at $(x, \log x)$ passes through $(0, -1 + \log x)$.

4. A SURPRISING RELATION BETWEEN EXPONENTIAL CURVES AND THE TRACTRIX. If a point Q moves on a straight line, if another point P not on this line pursues Q (that is, the motion of P is always directed toward Q), and if the distance from P to Q is constant, then P traces out a pursuit curve called a *tractrix*. All tangent segments from the tractrix to the line have constant length. An exponential curve has constant-length subtangents. Although the tractrix and exponential have been studied for centuries, apparently no one realized that they are related to one another. We show now that they are part of a new family of curves that we will describe presently.

Figure 8 (left) shows an arbitrary curve together with a fixed base line (shown here as the x-axis). At a general point P of this curve a tangent segment of length t cuts off a subtangent of length s along the base line. As before, we can form the tangent cluster by translating each tangent segment of length t parallel to itself so the point of tangency is brought to a common point O, as in Figure 8 (right). Let C denote the other endpoint of the tangent. As P moves along the given curve, point C traces out the curve defining the tangent cluster. We can also translate the subtangent of length s. These subtangents will be parallel to the given base line. One endpoint of the translated subtangent is at C. When point P moves along a tractrix, t is constant and C moves along a circle. When point P moves along an exponential, s is constant and C moves along a vertical line.

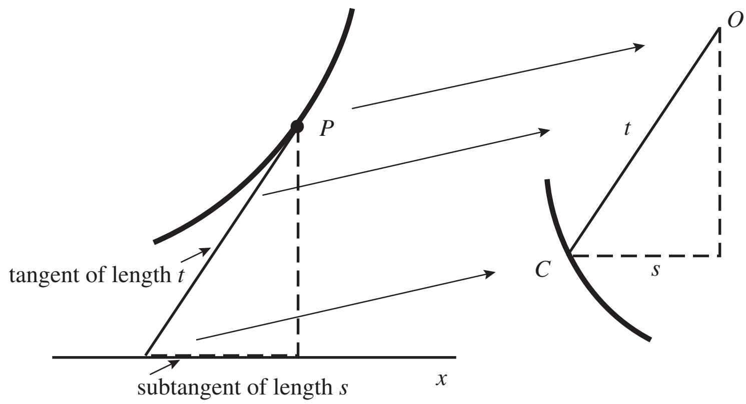

Figure 8. The tangent and subtangent to a general curve translated by the same amount.

Now suppose the original curve has the property that some linear combination of t and s has a constant value γ, say

$$\alpha t + \beta s = \gamma$$

for some choice of nonnegative α and β, not both zero. What is the path of C?

When $\beta = 0$, the tangent t is constant and C lies on a circle. When $\alpha = 0$, the subtangent s is constant and C lies on a straight line. Now we will show that, for general α and β, point C always lies on a conic section. Let's see why this is true.

If $\alpha \neq 0$, divide by α and rewrite the foregoing equation as

$$t = \varepsilon(d - s),$$

where $\varepsilon = \beta/\alpha$ and d is another constant. To show that C lies on a conic, refer to Figure 9. Use point O as a focus and take as directrix a line perpendicular to the subtangents at distance d from the focus and lying to the left of the vertical line through O. The quantity $(d - s)$ is the distance of C from the directrix, and t is the distance

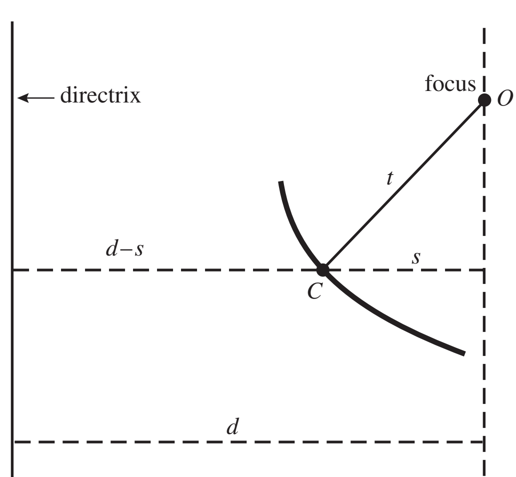

Figure 9. If $\alpha t + \beta s$ is constant, point C lies on a conic with eccentricity β/α.

of C from the focus. The equation $t = \varepsilon(d - s)$ states that the distance of C from the focus is ε times its distance from the directrix. Therefore C lies on a conic section with eccentricity ε. The conic is an ellipse, parabola, or hyperbola according as $0 < \varepsilon < 1$, $\varepsilon = 1$, or $\varepsilon > 1$. The limiting cases $\varepsilon = 0$ (i.e., $\beta = 0$) and ∞ (i.e., $\alpha = 0$) give a circle and vertical straight line, respectively. An intermediate case occurs when the original point P lies on a classical pursuit curve in which a fox running on a line is pursued by a dog (not on the line) having the same speed as the fox. In this case it is easily shown that $t + s$ is constant, so $\varepsilon = 1$ and the tangent cluster is a parabolic sector swept out by focal radii.

Thus, we have learned something new. The tractrix, the exponential, and the classical dog-fox-pursuit curve, which have been studied for hundreds of years, turn out to be special cases of a family of curves characterized by the equation $\alpha t + \beta s =$ constant. Cartesian equations for members of this family can be derived with the help of differential equations. However, these equations are not needed to determine the area of the tangent sweep of each member of the family. By Mamikon's theorem, the area of each tangent sweep is equal to that of the corresponding tangent cluster, which, in turn, is a sector of a conic section swept out by focal radii.

5. PROOF OF MAMIKON'S THEOREM.

Mamikon's theorem was proved in [**1**], but to make the proof more accessible, we present another version here.

Start with a smooth space curve Γ described by a position vector $\mathbf{X}(s)$, where s, the arclength function for the curve, varies over an interval, say $0 \le a \le s \le b$. The unit tangent vector to Γ is the derivative $d\mathbf{X}/ds$, which we denote by $\mathbf{T}(s)$. The derivative of the unit tangent is given by

$$\frac{d\mathbf{T}}{ds} = \kappa(s)\mathbf{N}(s),$$

where $\mathbf{N}(s)$ is the principal unit normal to Γ and $\kappa(s)$ is the curvature.

The curve Γ generates a surface S that can be represented by the vector parametric equation

$$\mathbf{y}(s, u) = \mathbf{X}(s) + u\mathbf{T}(s),$$

where u varies over an interval whose length can vary with s, say $0 \le u \le f(s)$. As the pair of parameters (u, s) varies over the ordinate set of the function f over the interval $[a, b]$, the surface S is swept out by tangent segments extending from the initial curve Γ to another curve described by the position vector $\mathbf{y}(s, f(s))$. Geometrically, S is a developable surface, that is, it can be rolled out flat on a plane without distortion. We refer to a surface S generated from a curve Γ in this fashion as a tangent sweep. The area of S is given by the double integral

$$a(S) = \int_a^b \int_0^{f(s)} \left\| \frac{\partial \mathbf{y}}{\partial s} \times \frac{\partial \mathbf{y}}{\partial u} \right\| du\, ds.$$

To calculate the integrand, note that

$$\frac{\partial \mathbf{y}}{\partial s} = \frac{d\mathbf{X}}{ds} + u\frac{d\mathbf{T}}{ds} = \mathbf{T}(s) + u\kappa(s)\mathbf{N}(s),$$

$$\frac{\partial \mathbf{y}}{\partial u} = \mathbf{T}(s), \qquad \frac{\partial \mathbf{y}}{\partial s} \times \frac{\partial \mathbf{y}}{\partial u} = u\kappa(s)\mathbf{N}(s) \times \mathbf{T}(s),$$

hence

$$\left\| \frac{\partial \mathbf{y}}{\partial s} \times \frac{\partial \mathbf{y}}{\partial u} \right\| = u\kappa(s),$$

since $\|\mathbf{N} \times \mathbf{T}\| = 1$. Therefore the integral for the area becomes

$$a(S) = \int_a^b \left(\int_0^{f(s)} u\,du \right) \kappa(s)\,ds = \frac{1}{2} \int_a^b f^2(s)\kappa(s)\,ds.$$

Next, imagine the arclength s expressed as a function of the angle φ between the tangent vector $\mathbf{T}$ and a fixed tangent line, say the tangent line corresponding to $s = a$. When s is expressed in terms of φ, the function $f(s)$ becomes a function of φ, and we write $f(s) = r(\varphi)$. On the surface S, φ is the angle between tangent geodesics, so the curvature κ is the rate of change of φ with respect to arclength, $\kappa = d\varphi/ds$. In the last integral we make a change of variable expressing s as a function of φ. Then $f^2(s) = r^2(\varphi)$, and $\kappa(s)\,ds = d\varphi$, so the integral becomes

$$a(S) = \frac{1}{2} \int_{\varphi_1}^{\varphi_2} r^2(\varphi)\,d\varphi, \tag{2}$$

where φ_1 and φ_2 are the initial and final angles of inclination corresponding to $s = a$ and $s = b$, respectively. Formula (2) shows that the area $a(S)$ does not depend explicitly on the arclength of Γ at all; it depends only on the angles φ_1 and φ_2. In fact, $a(S)$ is equal to the area of a plane radial set with polar coordinates (r, φ) satisfying $0 \le r \le r(\varphi)$ and $\varphi_1 \le \varphi \le \varphi_2$. ■

When (2) is reformulated in geometric terminology, it yields Mamikon's theorem in a form that has a strong intuitive flavor. If we translate each tangent segment of length $r(\varphi)$ parallel to itself so that each point of tangency is brought to a common vertex O, we obtain a portion of a conical surface that we call the tangent cluster of the curve Γ. Equation (2) gives us:

Mamikon's Theorem. *The area of a tangent sweep of a curve is equal to the area of its corresponding tangent cluster.*

The proof for special curves of the type described above follows from (2). The tangent cluster of S lies on a conical surface; this can be unrolled without distortion of area and the unrolled tangent cluster becomes a plane region whose area in polar coordinates is given by (2). Since this is also the area of S, the tangent sweep and its tangent cluster have equal areas. The theorem is also true for more general surfaces that can be decomposed into a sum or difference of a finite number of special surfaces of the type in the foregoing discussion. This will take care of tangent sweeps generated by piecewise smooth curves. For example, polygonal curves are treated in [**2**] and in [**3**]. It also takes care of curves with inflection points, where the sweeping tangent segments change their direction of rotation.

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