ARTICLES

Ellipse to Hyperbola: "With This String I Thee Wed"

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String construction for both ellipse and hyperbola The title was inspired by our modification of the well-known string construction for the ellipse. In FIGURE 1a a piece of string joins two fixed points (the foci of the ellipse), and the string is kept taut by a moving pencil that traces the ellipse. The bifocal property of the ellipse states that the sum of distances from pencil to foci is the constant length of the string.

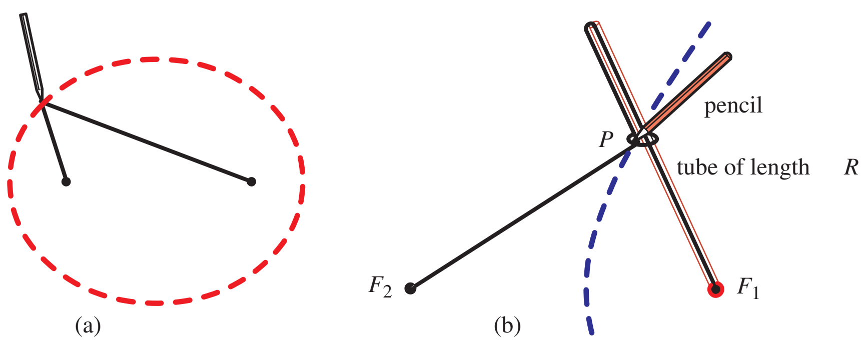

Figure 1 (a) String construction for the ellipse. (b) New mechanism for tracing a hyperbola. The pencil pushes inward along the outer edge of the tube.

The same string fastened to the same points can also be used to trace a hyperbolic arc with the same foci. How is this possible? The bifocal property of the hyperbola states that the *difference* of distances (longer minus shorter) from any point on the hyperbola to the foci is constant. Nevertheless, a slight modification of the string construction for the ellipse shows how to do it.

The points of intersection of an ellipse with the line through its foci are called its vertices. Take a thin rigid tube shorter than the string but longer than the distance from a focus to the nearest vertex. Pass part of the string through the tube and fasten the ends

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of the string to the foci as before. One end of the tube pivots at a focus, like one hand of a clock. The free end traces a circle that plays a crucial role in this paper. A pencil keeps the string taut by pushing it inward along the outer edge of the tube, as indicated in FIGURE 1b. If it pushes outward in the radial direction, the tube plays no role and the pencil traces part of the ellipse as in FIGURE 1a. But if it pushes inward as in FIGURE 1b, it traces a portion of a hyperbola lying inside the ellipse with the same foci, as in FIGURE 2a. This is easily verified by noting that the constant length c of the string is the sum of three distances in FIGURE 1b, the tube length R, plus $R - PF_1$ (the portion along the outside edge), plus focal distance PF_2. Therefore $PF_2 - PF_1 = c - 2R$, a constant.

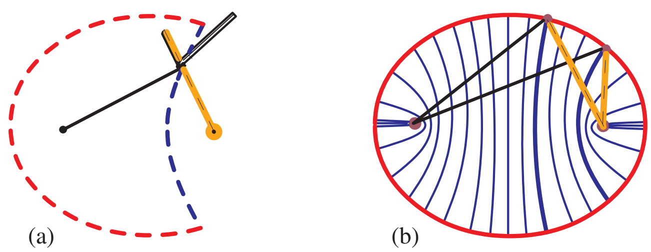

Figure 2 (a) The hyperbolic arc is inside the ellipse with the same foci. (b) If the length of the tube is varied, the pencil traces arcs of all confocal hyperbolas.

By varying the length of the tube you can draw an entire family of confocal hyperbolic arcs (FIGURE 2b). Because these arcs are confocal with the ellipse, they intersect it orthogonally. One of the arcs so constructed is the perpendicular bisector of the segment joining the foci.

Contents of this paper The string mechanism that weds the ellipse and hyperbola leads in a natural way to a generalization of the classical bifocal property, in which each focus is replaced by a circle, called a *focal circle*, centered at that focus. Focal circles extend the string construction by using two tubes, each pivoted at a focus; each free end traces a focal circle. Theorem 1 reveals that each of the sum and difference of distances to the focal circles can be constant on both the ellipse and hyperbola. Special pairs of focal circles, called *circular directrices*, are then introduced. Those familiar with paper-folding activities for constructing an ellipse or hyperbola using a circle as a guide, will be pleased to learn that the guiding circle is, in fact, a circular directrix. This is followed by an extended bifocal property for the ellipse and hyperbola, a converse to Theorem 1.

Although a parabola has only one focus, the extended bifocal properties of the ellipse and hyperbola can be transferred to a parabola by moving one focus to ∞. In the limit, a circular directrix centered at the moving focus becomes the classical directrix of the parabola. An application is also given to a pursuit problem involving conics.

Focal circles for ellipse and hyperbola

FIGURE 3 shows a string mechanism generalizing that in FIGURE 1b for tracing both elliptic and hyperbolic arcs with the same foci. This involves two tubes, each pivoting around a focus. The free end of each tube traces a circle that we call a *focal circle*. The focal circles may or may not intersect, and one of them might lie inside the other. The example in FIGURE 3 shows them intersecting. Join the foci with a string of constant

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length which passes through the two tubes. A new feature, not needed in FIGURE 1b, is the introduction of a ring to insure that the pencil keeps the string taut at the intersection of the radial directions. The four diagrams in FIGURE 3 show how the mechanism works in different parts of the plane determined by the intersecting focal circles.

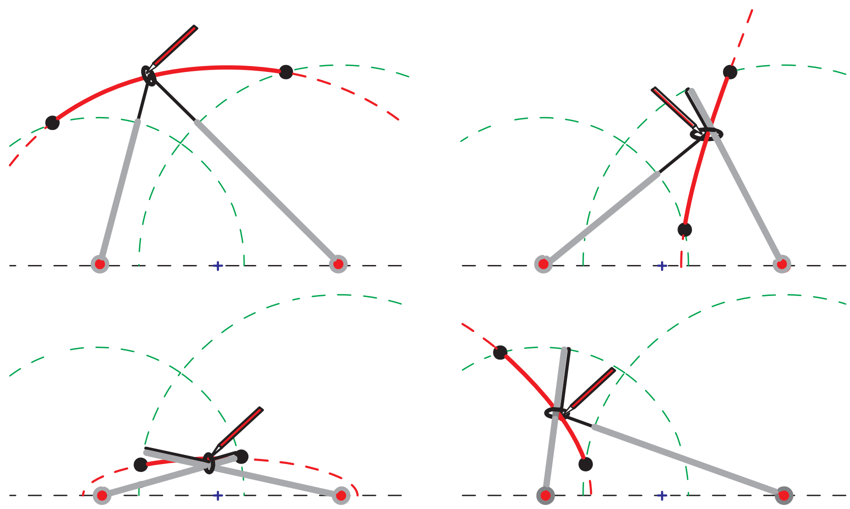

Figure 3 String mechanism involving two tubes. A wedding ring keeps the string taut at the intersection of the radial directions.

What is the locus traced by continuous motion of this string mechanism?

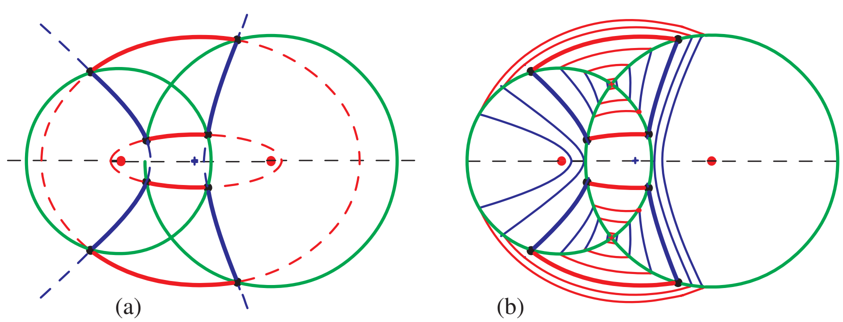

Figure 4 (a) A curvilinear trapezoid and its mirror image, each traced by one continuous motion of the two-tube string mechanism. (b) A family of trapezoids obtained by varying the length of the portion of the string outside the tubes.

The result, which may seem surprising, is a curvilinear 'trapezoid' bounded by elliptic and hyperbolic arcs, as shown in FIGURE 4a. FIGURE 4b shows a family of curvilinear trapezoids obtained by varying the length of the portion of the string outside the tubes.

To analyze the situation more precisely, refer to FIGURE 5 which shows two distinct points F_1 and F_2 that will serve as foci for an ellipse or a hyperbola. Draw two coplanar circles C_1 and C_2, which are the focal circles, centered at the foci with respective radii $R_1 \geq 0$ and $R_2 \geq 0$.

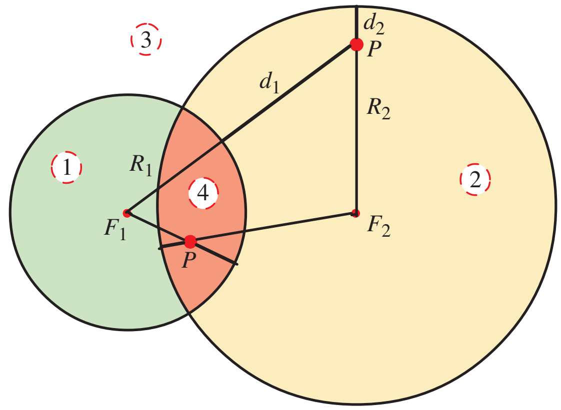

Figure 5 Two focal circles that divide the plane into four regions.

The example in FIGURE 5 shows two intersecting focal circles that divide the plane into four regions: region 1 inside C_1 and outside C_2, region 2 inside C_2 and outside C_1, region 3 outside both C_1 and C_2, and region 4 inside both C_1 and C_2. In some cases, one of regions 1, 2 or 4 may be empty.

FIGURE 4a shows a curvilinear trapezoid and its mirror image, each of which can be traced by the string mechanism in one continuous motion through all four regions in FIGURE 5. The upper trapezoid has two lower vertices on the boundary of region 4, and two upper vertices on the boundary of region 3. Place the pencil at the lower right vertex on circle C_1, moving it through region 4 to the lower left vertex on circle C_2. As we show later, this traces an arc of an ellipse (the lower edge of the trapezoid). Now continue the motion in region 1 to trace a hyperbolic arc (the left edge of the trapezoid), and then in region 3 to trace another elliptical arc (the upper edge of the trapezoid). Finally, return to the starting point by tracing another hyperbolic arc in region 2 (the right edge of the trapezoid). Theorem 1a will show that the length d of the portion of the string outside the tubes is the same constant on each edge of the trapezoid. By changing the value of d we obtain an entire family of trapezoids, as depicted in FIGURE 4b. As d shrinks to 0 the trapezoid becomes a point of intersection of the focal circles.

Two locus properties relating the ellipse and hyperbola

This section introduces two new and surprising locus properties relating the ellipse and hyperbola. Refer to the focal circles in FIGURE 5. Choose any point P in the plane of the circles, and let f_1 be the distance from P to focus F_1, and f_2 the distance from P to focus F_2. Also, let d_1, d_2 be the respective shortest distances from P to focal circles C_1 and C_2, each measured radially, so that $d = d_1 + d_2$ is the length of the portion of the string outside the tubes in the string mechanism. FIGURE 5 shows two choices of P, one in region 2, the other in region 4. We note that the following relations hold in FIGURE 5:

$$\text{In region 1, } d_1 = R_1 - f_1 \text{ and } d_2 = f_2 - R_2. \tag{1}$$

$$\text{In region 2, } d_1 = f_1 - R_1 \text{ and } d_2 = R_2 - f_2. \tag{2}$$

$$\text{In region 3, } d_1 = f_1 - R_1 \text{ and } d_2 = f_2 - R_2. \tag{3}$$

$$\text{In region 4, } d_1 = R_1 - f_1 \text{ and } d_2 = R_2 - f_2. \tag{4}$$

By adding d_1 and d_2 in each region we obtain:

LEMMA 1.

(1) *If P is in region* 1, *then* $d_1 + d_2 = (f_2 - f_1) - (R_2 - R_1)$.
(2) *If P is in region* 2, *then* $d_1 + d_2 = (f_1 - f_2) - (R_1 - R_2)$.
(3) *If P is in region* 3, *then* $d_1 + d_2 = (f_1 + f_2) - (R_1 + R_2)$.
(4) *If P is in region* 4, *then* $d_1 + d_2 = (R_1 + R_2) - (f_1 + f_2)$.

When $d_1 + d_2$ is constant, Lemma 1 reveals the following information about the curves traced by the string mechanism:

In region 1, $f_2 - f_1$ is constant and P traces part of a hyperbola with foci F_1 and F_2. In FIGURE 4a, this part is shown as two solid arcs on the left branch of this hyperbola.

In region 2, $f_1 - f_2$ is a different constant and P traces part of a different hyperbola with the same foci. In FIGURE 4a, this part is shown as two solid arcs on the right branch of the second hyperbola.

In region 3, $f_1 + f_2 = R_1 + R_2 + d_1 + d_2 = c$, the length of the string, and P traces part of an ellipse, shown in FIGURE 4a as two solid elliptical arcs.

In region 4, the constant focal sum $f_1 + f_2$ differs from that in region 3, and P traces two solid arcs of the smaller ellipse shown in FIGURE 4a.

By subtracting distances d_1 and d_2 in each region, we obtain:

LEMMA 2.

(1) *If P is in region* 1, *then* $d_2 - d_1 = (f_1 + f_2) - (R_1 + R_2)$.
(2) *If P is in region* 2, *then* $d_1 - d_2 = (f_1 + f_2) - (R_1 + R_2)$.
(3) *If P is in region* 3, *then* $d_1 - d_2 = (f_1 - f_2) - (R_1 - R_2)$.
(4) *If P is in region* 4, *then* $d_2 - d_1 = (f_1 - f_2) - (R_1 - R_2)$.

When $|d_1 - d_2|$ is constant, Lemma 2 reveals the following information about the curves traced by the string mechanism:

In regions 1 and 2, the focal sum $f_1 + f_2 = R_1 + R_2 + |d_1 - d_2|$ is constant, so P traces an elliptical arc. Each of these arcs, shown dashed in FIGURE 4a, is a continuation of a corresponding solid elliptical arc in FIGURE 4a, because their focal sums agree at those points where the arcs intersect the focal circles.

A similar analysis shows that each dashed hyperbolic arc in regions 3 and 4 of FIGURE 4a is a continuation of a corresponding solid hyperbolic arc in regions 1 and 2.

Thus, from Lemmas 1 and 2 we deduce:

THEOREM 1.

(a) *The locus of points P such that $d_1 + d_2$ is constant is part of an ellipse or a hyperbola.*
(b) *The locus of points P such that $|d_1 - d_2|$ is constant is part of an ellipse or a hyperbola.*

Proof. Part (a) follows from Lemma 1, and part (b) from Lemma 2. ■

Theorem 1 uncovers the surprising fact that use of focal circles allows each of $d_1 + d_2$ and $|d_1 - d_2|$ to be constant on both the ellipse and hyperbola.

Circular directrices for the ellipse and hyperbola Unlike central conics (ellipse and hyperbola), which have two foci, a parabola has only one focus F. It can be described as the locus of a point P that moves in a plane with its focal distance PF always equal to its distance PD from a fixed line D, called its directrix. Now we introduce an analogous equidistant property for central conics, using two special focal circles with the property that *from each point of the central conic, the shortest distances to the two focal circles are equal.* If such focal circles exist, we call them *circular directrices*. Now we will show that Lemma 2 implies that they *do* exist and tells us how to determine them.

A point P is equidistant (equal shortest distances) from the two focal circles if, and only if, $d_1 = d_2$. According to Lemma 2, this happens in regions 1 and 2 of FIGURE 5 when $R_1 + R_2 = f_1 + f_2$, and the same occurs in regions 3 and 4 when $R_1 - R_2 = f_1 - f_2$.

On an ellipse, the constant sum $f_1 + f_2$ represents the length of the major axis of the ellipse, while on a hyperbola the constant difference $|f_1 - f_2|$ represents the length of the transverse axis. This gives us the following:

DESCRIPTION OF CIRCULAR DIRECTRICES. *Any two focal circles with sum of radii $R_1 + R_2 = f_1 + f_2$ serve as a pair of circular directrices for the ellipse; any two focal circles with $|R_1 - R_2| = |f_1 - f_2|$ serve as a pair of circular directrices for the hyperbola.*

Each central conic has *infinitely many pairs* of circular directrices. FIGURE 6 shows three pairs of circular directrices for a given ellipse. In (a), $R_1 = 0$, C_1 becomes focus F_1, and $R_2 = f_1 + f_2$, so the entire ellipse lies inside focal circle C_2, hence in region 2. Each point on the ellipse is equidistant from focus F_1 and from this particular focal circle C_2, as depicted in FIGURE 6a. This circular directrix C_2 is used in standard paper-folding constructions for the ellipse. It was also used by Feynman [**2**, p. 152] in his geometric treatment of Kepler's laws of planetary motion.

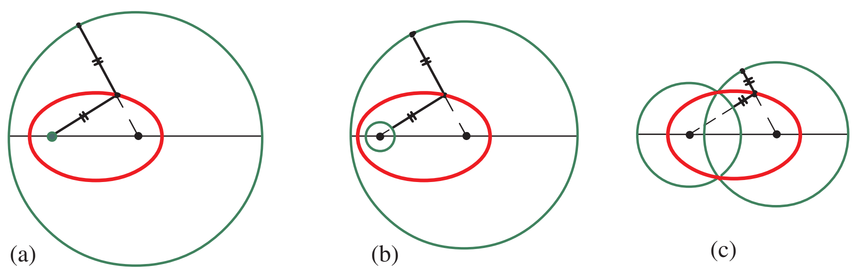

Figure 6 An ellipse and three pairs of its circular directrices. In (a), $R_1 = 0$ and $R_2 = f_1 + f_2$. In (b) and (c), $R_1 > 0$ and $R_2 = f_1 + f_2 - R_1$. For each P on the ellipse, the shortest distances d_1 and d_2 to the focal circles are equal.

In FIGURES 6b and 6c, both circular directrices have positive radii, but the sum of the radii is constant, so an increase in R_1 results in a corresponding decrease in R_2. In FIGURE 6b, the entire ellipse is in region 2, but in FIGURE 6c part of the ellipse is in region 2 and the remaining part in region 1.

FIGURE 7 shows three pairs of circular directrices for one given branch of a hyperbola. In FIGURE 7a, $R_1 = 0$ and $R_2 = f_2 - f_1$. This circular directrix is used in standard paper-folding constructions for the hyperbola. In FIGURES 7b and 7c, $R_1 > 0$ and $R_2 = f_2 - f_1 + R_1$, so an increase in R_1 results in a corresponding increase in R_2.

One may very well ask *"What about the other branch of the hyperbola?"*

FIGURE 8 shows both branches and reveals something new. From any point P there are *two* distances to each focal circle, the *shortest* distances, which we have denoted by d_1 and d_2, and the *longest* distances, which we denote by D_1 and D_2. The difference $D_i - d_i$ is $2R_i$, the diameter of focal circle C_i. The longest distance D_2 plays a role on the second branch. FIGURE 8a shows both branches of the hyperbola in FIGURE

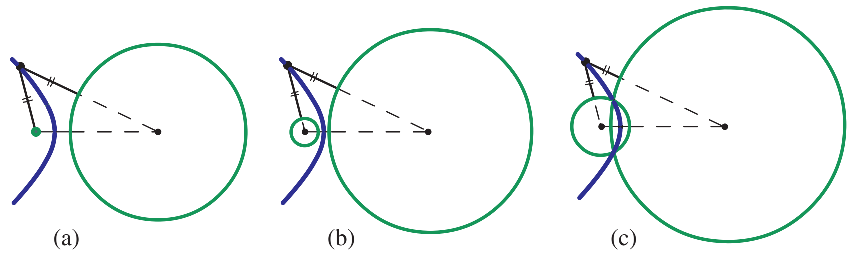

Figure 7 One branch of a hyperbola and three pairs of its circular directrices. In (a), $R_1 = 0$, $R_2 = f_2 - f_1$. In (b) and (c), $R_1 > 0$, $R_2 = f_2 - f_1 + R_1$. For each P on the hyperbola, the shortest distances d_1 and d_2 to the focal circles are equal.

7a, labeled as H_1. Here we have $d_1 = D_2$ on the second branch. In other words, the shortest distance to C_1 is equal to the longest distance to C_2. In this case, $C_1 = F_1$ because $R_1 = 0$.

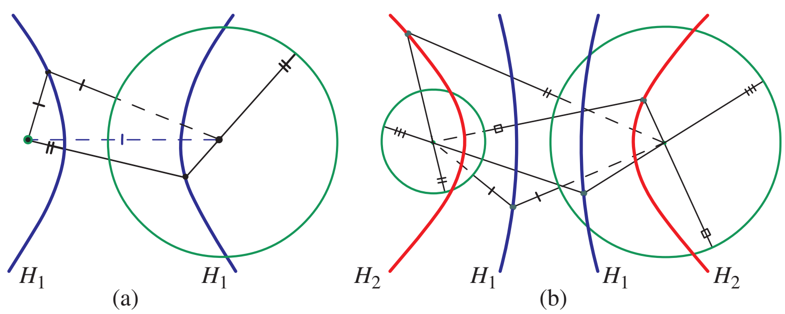

Figure 8 (a) Both branches of hyperbola in FIGURE 7a. For each P on the right branch the shortest distance d_1 is equal to the longest distance D_2. (b) Two confocal hyperbolas H_1 and H_2 with transverse axes of different lengths.

But when $R_1 > 0$ and $R_2 = f_2 - f_1 + R_1$, a new phenomenon occurs. A second hyperbola comes into play with the same foci but with a different transverse axis, as shown in FIGURE 8b. Let H_1 denote the hyperbola with the shorter transverse axis, and H_2 the one with the longer. Each point on the left branch of H_1 has $d_1 = d_2$, as in FIGURE 7, but each point on the right branch of H_1 has $D_1 = D_2$. On the left branch of H_2 we have $D_1 = d_2$, and on the right branch of H_2 we have $d_1 = D_2$, as indicated by tick marks in FIGURE 8b. Circular directrices play a deeper role than indicated in FIGURE 6.

This is illustrated further in FIGURE 9a, which can be thought of as a continuation of FIGURE 8b. As radius R_2 increases, the asymptotes of hyperbola H_2 become more and more horizontal until R_2 reaches a critical value for which H_2 degenerates to a pair of rays emanating from the foci. For points on the degenerate hyperbola, $|f_2 - f_1|$ is the distance between the foci, which is also $f_1 + f_2$, the sum of focal distances from points on the line segment joining the foci. This segment is a degenerate ellipse. As R_2 increases beyond the critical value and the circular directrices C_1 and C_2 intersect as shown in FIGURE 9a, hyperbola H_2 in FIGURE 8b is replaced by a confocal ellipse E_2 on which $d_1 = d_2$. On the left branch of H_1 we have $d_1 = d_2$, and on its right branch we have $D_1 = D_2$, as in FIGURE 8b. As radius R_2 increases further, so that C_2 contains C_1 in its interior, as in FIGURE 9b, hyperbola H_1 also degenerates and is replaced by a second confocal ellipse E_1 on which $d_1 = d_2$.

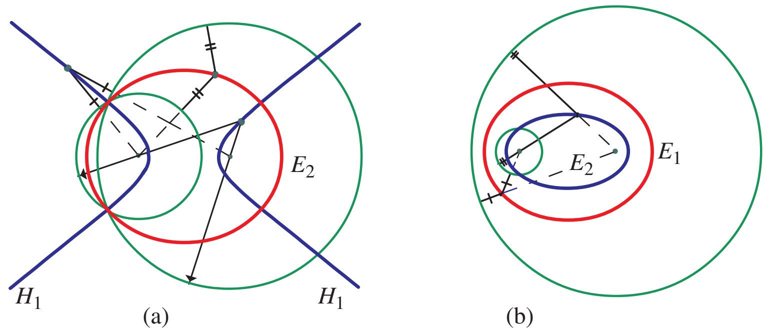

Figure 9 (a) Hyperbola H_2 in FIGURE 8b is replaced by ellipse E_2. (b) Hyperbolas H_1 and H_2 in FIGURE 8b are replaced by ellipses E_1 and E_2, respectively.

Extended bifocal property of the ellipse and hyperbola

The next theorem provides an extended bifocal property which has the same form for the ellipse and for the hyperbola. It is stated in terms of the shortest distances d_1, d_2 to the focal circles. Recall that the sum of focal distances $f_1 + f_2$ from any point on a ellipse to its foci is a constant equal to the length of the major axis, which we denote by A. On a hyperbola, the difference $|f_1 - f_2|$ of focal distances is another constant equal to the length of the transverse axis, which we denote by B. On the left branch (enclosing focus F_1) we have $f_2 - f_1 = B$, and on the other branch we have $f_1 - f_2 = B$.

THEOREM 2.

(a) *Given an ellipse with major axis of length A, and given two focal circles C_1, C_2 of radii R_1, R_2. Let $d = R_1 + R_2 - A$. Then each point on the ellipse satisfies*

$$d_1 + d_2 = |d| \tag{5}$$

or

$$|d_1 - d_2| = |d|. \tag{6}$$

(b) *Given a hyperbola with transverse axis of length B, and given the same focal circles as in (a). Let $d' = B - (R_1 + R_2)$. Then each point on the left branch of the hyperbola satisfies*

$$d_1 + d_2 = |d'| + 2R_1 \tag{7}$$

or

$$|d_1 - d_2| = |d'| + 2R_1. \tag{8}$$

Each point on the right branch satisfies

$$d_1 + d_2 = |d'| + 2R_2 \tag{9}$$

or

$$|d_1 - d_2| = |d'| + 2R_2. \tag{10}$$

If $R_1 = R_2$, then on both branches we have $d_1 + d_2 = B$ or $|d_1 - d_2| = B$.

Proof of (a). We consider two cases, depending on the algebraic sign of d.

Case 1. $d \leq 0$, so $R_1 + R_2 \leq A$. If both focal circles lie inside the ellipse, then $d_1 = f_1 - R_1$ and $d_2 = f_2 - R_2$, hence $d_1 + d_2 = f_1 + f_2 - (R_1 + R_2) = A - (R_1 + R_2) = -d = |d|$, so (5) holds on the entire ellipse.

If the ellipse intersects a focal circle, say C_1, then at the point of intersection we have $d_1 = 0$, $f_1 = R_1$, $d_2 = f_2 - R_2 = A - f_1 - R_2 = A - (R_1 + R_2) = -d = |d|$, hence both (5) and (6) are satisfied at this point. But Lemma 2 shows that for this value of d, (6) holds for every point of the ellipse in regions 1 or 2. A similar argument works if the ellipse intersects focal circle C_2. This proves (a) in Case 1.

Case 2. $d > 0$, so $R_1 + R_2 > A$. Now the focal circles intersect each other and also intersect the ellipse. At a point where C_1 intersects the ellipse we have $d_1 = 0$, $f_1 = R_1$, $d_2 = f_2 - R_2 = A - f_1 - R_2 = A - (R_1 + R_2) = -d$, hence $d_1 - d_2 = d = |d|$ at the point of intersection. But Lemma 2(1) shows that $d_1 - d_2 = d$ for every point of the ellipse in region 1, and that $d_2 - d_1 = d$ in region 2. Also, Lemma 1(4) shows that $d_1 + d_2 = d$ for every point of the ellipse in region 4. This proves (a) in Case 2. ■

Proof of (b). On a hyperbola $|f_1 - f_2|$ is constant, so $B = |f_1 - f_2|$. Again we consider two cases, depending on the relation between B and $R_1 + R_2$.

Case 1. $B \geq R_1 + R_2$. In this case the focal circles do not intersect, and at most one of them can intersect the hyperbola. If neither focal circle intersects the hyperbola, then $f_1 = R_1 + d_1$ and $f_2 = R_2 + d_2$, hence $d_1 - d_2 = f_1 - f_2 + R_2 - R_1$, which is the same as $d_2 - d_1 = f_2 - f_1 + R_1 - R_2$. On the left branch, $f_1 < f_2$, so $f_2 - f_1 = B$ and $d_2 - d_1 = B + R_1 - R_2 = B - (R_1 + R_2) + 2R_1$, so (8) is satisfied everywhere on this branch.

On the right branch, $f_2 < f_1$, so $f_1 - f_2 = B$ and $d_1 - d_2 = B + R_2 - R_1 = B - (R_1 + R_2) + 2R_2$, so (10) is satisfied everywhere on this branch.

Now suppose that one focal circle, say C_1, intersects the hyperbola. At a point of intersection we have $d_1 = 0$, $R_1 = f_1$ so $d_2 - d_1 = f_2 - R_2 = f_2 - f_1 + R_1 - R_2 = B + R_1 - R_2 = B - (R_1 + R_2) + 2R_1$. By Lemma 2(3), $d_2 - d_1$ has the same value everywhere in region 3, hence (8) holds everywhere in region 3. Now by Lemma 1(1), in region 1 we have $d_1 + d_2 = f_2 - f_1 - R_2 + R_1 = B - (R_1 + R_2) + 2R_1$, so (7) holds everywhere in region 1. Therefore either (7) or (8) holds on the left branch, whereas (10) holds everywhere on the right branch. If, however, focal circle C_2 intersects the hyperbola, then the same type of argument shows that either (9) or (10) holds on the right branch, and (8) holds everywhere on the left branch. This proves (b) in Case 1.

Case 2. $B < R_1 + R_2$. In this case the focal circles overlap and each intersects the hyperbola. The same type of argument used for Case 1 shows that, on the left branch, (7) holds in regions 3 and 4, and (8) holds in region 1. Similarly, on the right branch, (9) holds in regions 3 and 4, and (10) holds in region 2. ■

Bifocal properties transferred to the parabola

The extended bifocal properties of the central conics were obtained by replacing each focus by a focal circle. Although the parabola has only one focus, we can transfer the extended bifocal properties to the parabola by keeping one focal circle fixed and moving the second focus to ∞, allowing the radius of the second focal circle to increase without bound. The second focal circle now becomes a line perpendicular to the focal axis, which we call a *floating focal line*. The central conic becomes a parabola whose focus is the center of the fixed focal circle, and whose directrix is parallel to the floating focal line. As expected, the bifocal properties of the central conics can be transferred to the focal circle and the floating focal line.

This process is consistent with the geometric definition of conics as sections of a cone. Recall that a plane cutting one nappe of a right circular cone produces an ellipse.

As the plane is tilted to become nearly parallel to a generator of the cone, the ellipse becomes more elongated, and when the cutting plane is parallel to a generator the intersection becomes a parabola. Tilt the plane even further so it cuts both nappes, and the intersection is a hyperbola. Thus, as a section of a cone, the parabola is a transition between the ellipse and hyperbola, so it's not surprising that properties of the parabola can be obtained as limiting cases of those of a central conic.

First we introduce the parabolic version of circular directrices. For central conics, circular directrices occur in pairs, each a special focal circle. In the parabolic version, one of the circular directrices is replaced by a limiting line called a floating directrix.

Pairs of circular and floating directrices for the parabola Recall that a parabola has only one focus F, and is the locus of a point P that moves in a plane with its focal distance PF always equal to its distance PD from a fixed line D, called its directrix (FIGURE 10a). We call D the *linear directrix* of the parabola, to distinguish it from *circular directrices*, which we define as follows. Any line L parallel to directrix D we call a *floating focal line*. In FIGUREs 10b and 10c, L is between F and D. Let R denote the distance between L and D. Then the focal circle $C(R)$ of radius R and center F is called a *circular directrix* for the parabola, *relative to* L. In this context, line L is also called a *floating directrix* corresponding to the circular directrix. This terminology was chosen because for every point P on the parabola we have $d_C = d_L$, where d_C is the shortest distance from P to focal circle $C(R)$, and d_L is the distance from P to L. This common distance is $PF - R$, as shown in FIGURES 10b and 10c. *When the radius of C is zero, circular directrix C becomes the focus of the parabola, and floating directrix L becomes its classical directrix D,* as in FIGURE 10a.

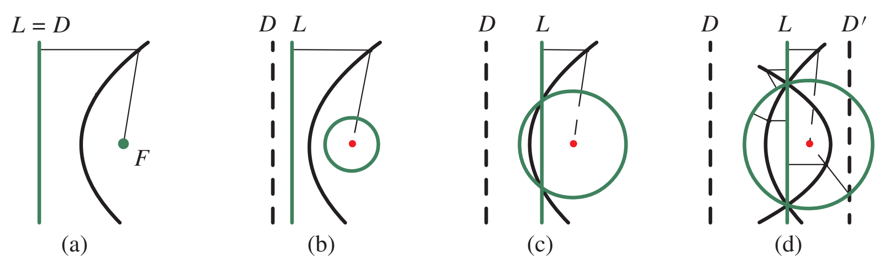

Figure 10 (a) Focus and directrix of a parabola. In (b)–(d), circular directrix and floating directrix. In (d), two intersecting confocal parabolas with linear directrices D and D', but with the same circular directrix and floating directrix.

As expected, each circular directrix of a parabola can be obtained as the limiting case of a circular directrix of a central conic by sending one of the foci to ∞. To illustrate, begin with an ellipse and two circular directrices C_1, C_2, as in FIGURE 6b or 6c, where $d_1 = d_2$ for each point on the ellipse. Let Q be the point where circle C_2 intersects the focal axis. Keep F_1, R_1 and Q fixed, and move focus F_2 along the focal axis arbitrarily far away, so that $R_2 \to \infty$. Then, the limiting circle C_2 becomes a line L through Q perpendicular to the focal axis. The radial distance d_2 becomes d_L, the distance from P to L, and the ellipse becomes a limit curve with the property that $d_1 = d_L$ at each of its points. This limit curve is, in fact, a parabola with focus F_1 and linear directrix D, whose distance from L is R_1, because each of its points is equidistant from F_1 and D. The circular directrix C_1 for the ellipse is now a circular directrix for the parabola with L as its floating directrix. The parabola opens to the right, as in FIGURES 10b and 10c. The initial choice of Q determines the position of the floating directrix L.

We can arrive at the same circular directrix and the same floating directrix L by starting with the left branch of the hyperbola shown in FIGURE 7, keeping F_1, R_1 and Q fixed as before, and letting $R_2 \to \infty$. If Q is between F_1 and the vertex of the left branch, as in FIGURE 7c, the limit curve is a parabola that opens to the left and intersects the first parabola, as shown by the example in FIGURE 10d, with its linear directrix D' parallel to L. Both parabolas intersect the floating directrix L and the circular directrix at the same points.

Parabolic version of the extended bifocal properties The extended bifocal properties of central conics in Theorems 1 and 2 have counterparts for the parabola. They can be obtained by starting with two focal circles C_1 and C_2 of a central conic, and letting the radius of one of them, say R_2, go to ∞, keeping F_1, R_1, and Q fixed, as was done earlier. The limiting C_2 becomes a floating focal line L through Q perpendicular to the focal axis, and the limiting central conic becomes a parabola with focus F_1 and focal circle C_1. The new properties relate C_1 and L.

FIGURE 11a shows what happens to FIGURE 5 in this limiting case, and FIGURE 11b shows how the four central conics in FIGURE 4a become four parabolas. On the solid portions of the parabolas, the sum of distances $d_1 + d_L$ to the focal circle and the floating focal line is constant, just as on the corresponding ellipses and hyperbolas in FIGURE 4a. On the dashed portions, the absolute difference $|d_1 - d_L|$ is constant, just as on the corresponding central conics in FIGURE 4a. FIGURE 11b also illustrates the following parabolic counterparts of Theorems 1 and 2, whose proofs are omitted.

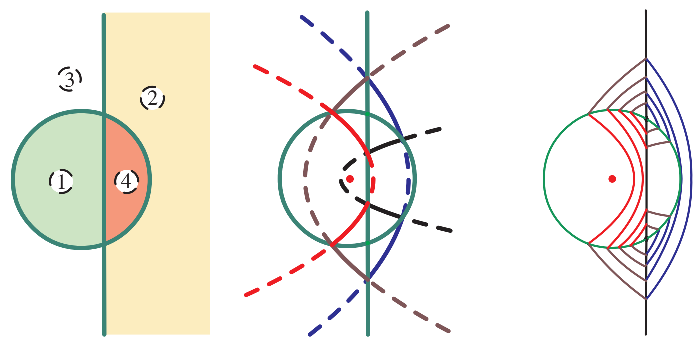

Figure 11 (a) Four regions formed by a focal circle and a coplanar line. (b) Limiting case of FIGURE 4a when the ellipses and hyperbolas become four parabolas. (c) Family of parabolic trapezoids obtained as limiting case of FIGURE 4b.

THEOREM 3. *Given a circle C with center at F, and a coplanar line L. If P is in the plane of C and L, let d_C and d_L denote the shortest distances from P to C and L, respectively. Then the locus of points P such that either the sum $d_C + d_L$ or the absolute difference $|d_C - d_L|$ is constant is part of a parabola with focus F and directrix parallel to L.*

THEOREM 4. *Given a parabola with a focal circle C, and given any line L parallel to its directrix D, whose distance from D is the radius of C. Then either the sum $d_C + d_L$ or the absolute difference $|d_C - d_L|$ is constant.*

String construction for the parabola When focal circle C has radius 0, the property that $d_C + d_L$ is constant reduces to $d_F + d_L$ is constant, and leads to a string construction for the parabola, illustrated in FIGURE 12a. One endpoint of a string of constant length is fastened to a fixed point, but the other end is attached to a small ring that slides freely along a rigid rod (a fixed line) that may or may not pass through the fixed point. Again, the string is kept taut by a pencil that moves so that the sum of distances from the pencil to the fixed point and to the fixed line is the constant length of the string. The pencil traces a portion of a parabola with the fixed point as its focus. A second parabola with the same focus can be drawn by placing the pencil on the other side of the fixed line, as indicated in FIGURE 12a. Kepler [**3**, p. 110] devised a similar string construction for the parabola that does not use a ring and produces only one of the two parabolas. Our construction for two confocal parabolas is justified by the diagram in FIGURE 12b, which shows the common circular directrix of the two parabolas with the fixed line as floating directrix.

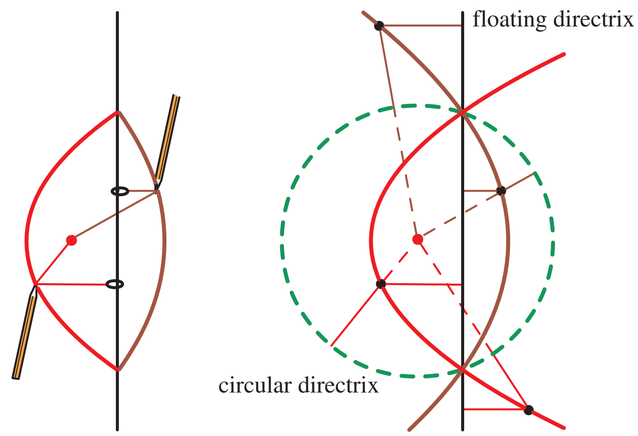

Figure 12 (a) String construction that gives two confocal parabolas, one on either side of the fixed line. (b) Justification of construction, using the common circular directrix of the two parabolas, with the corresponding floating directrix.

FIGURE 13a shows an equivalent form of the string construction in FIGURE 12a with a focal circle C of radius $R > 0$ centered at F. Then $d_C = d_F - R$, and the constant sum $d_F + d_L$ is replaced by $d_C + d_L = d_F + d_L - R$, another constant. The small ring that moves freely around the rigid boundary of the focal circle allows the pencil to trace the parabolas in FIGURE 12. Similarly, the two tubes and the single ring in the string mechanism of FIGURE 3 could be replaced by two small rings (illustrated by two examples in FIGURE 13b) that move freely around the two focal circles, with

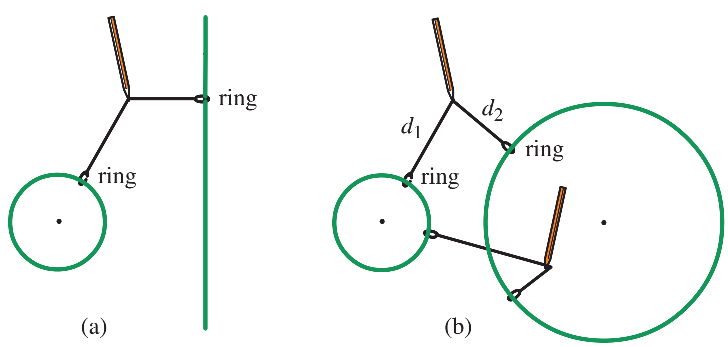

Figure 13 Alternative string construction (a) for parabolas, using a focal circle, and (b) for central conics, using two focal circles.

one end of the string attached to each ring. The pencil keeps the string taut so that the two portions of the string are in the appropriate radial directions.

Application of Theorem 3 to a pursuit problem A classical pursuit problem involves an aircraft flying at constant speed $v > 0$ from a given point A toward a fixed base F (FIGURE 14a). Because visibility is limited, an automatic pilot always aims the aircraft toward F. Ordinarily, the path would be along a straight line from A to F. However, a steady north wind with constant speed w forces the aircraft off course, so its trajectory is along a curved path which depends on the ratio of the speeds v and w. The problem is to determine this path.

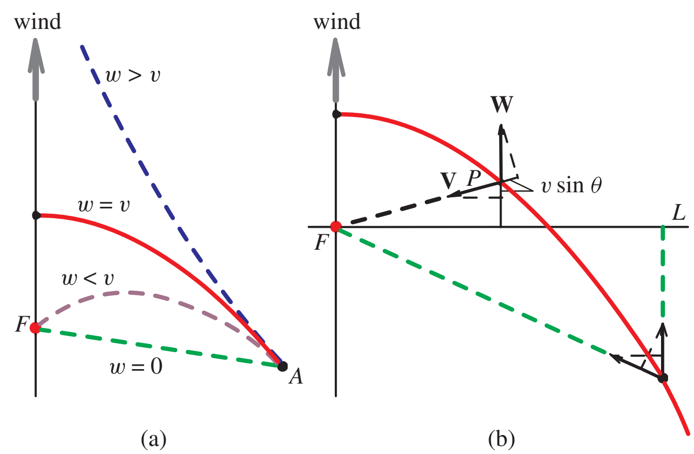

Figure 14 (a) Qualitative shape of trajectory depends on speeds v and w. (b) When $v = w$, the sum $d_F + d_L$ is constant above L, whereas $d_F - d_L$ is constant below L, hence the trajectory is a parabola with focus F.

If $w > v$, the craft cannot overcome the influence of the wind and moves further away from the base, approaching asymptotically the line due north from F. But if $v > w$, the aircraft overcomes the influence of the wind and returns to F along a curved path. These two solutions, which seem intuitively reasonable, can also be verified analytically by solving a suitable differential equation. The dashed curves in FIGURE 14a indicate the qualitative nature of the solutions. The line segment from A to F shows the path when $w = 0$.

The case of interest for us is when $w = v$. In this case the solution of the differential equation is part of a parabola, shown as the solid curve in FIGURE 14a. Point F is the focus of this parabola. The aircraft moves along the parabola until it is due north of F at which point it remains stationary because the effect of its speed and that of the wind cancel each other. We shall obtain this solution by applying Theorem 3.

Choose a line L through F perpendicular to the wind direction, as in FIGURE 14b. We regard L as a focal line, and let F serve as a focal point. Let P denote a general point on the path of the aircraft, and let d_F and d_L denote its distances from F and L, respectively, as indicated in FIGURE 14b. Line L divides the trajectory into two parts, one above L and one below. We will show that the sum $d_F + d_L$ is constant when P is above L, and that the difference $d_F - d_L$ is the same constant when P is below L. By Theorem 3 with $C = F$, this will prove that the path is a parabola with focus F.

Suppose P is above L. Let θ denote the angle between L and the line joining F to P. In general, point P moves along a tangent vector to the path with velocity $\mathbf{V} + \mathbf{W}$, the resultant of two vectors $\mathbf{V}$ and $\mathbf{W}$ of lengths $v = |\mathbf{V}|$ and $w = |\mathbf{W}|$. We are considering the case in which $w = v$. Vector $\mathbf{W}$, in the direction of the wind, acts to increase d_L at the time rate v. But $\mathbf{V}$ acts to decrease d_L by a component of magnitude $v \sin\theta$ opposite to $\mathbf{W}$. Hence the resultant $\mathbf{V} + \mathbf{W}$ has a component in the direction of $\mathbf{W}$ equal to $v - v \sin\theta$, which represents the time rate of change of d_L. Similarly, the component of the resultant in the direction of $\mathbf{V}$ is $v \sin\theta - v$, which represents the time rate of change of d_F. Therefore the time rate of change of the sum $d_F + d_L$ is zero, hence $d_F + d_L$ is constant. This constant is d_F when $d_L = 0$, and is the distance from F to the point where L intersects the trajectory.

When P is below L the analysis is similar, except that both $\mathbf{V}$ and $\mathbf{W}$ act to decrease d_L so the resultant $\mathbf{V} + \mathbf{W}$ has a component in the direction of $\mathbf{W}$ of magnitude $v + v \sin\theta$, whose negative is the time rate of change of d_L, and a component in the direction of $\mathbf{V}$ of the same magnitude, whose negative is the time rate of change of d_F. Therefore the time rate of change of the difference $d_F - d_L$ is zero, hence $d_F - d_L$ is constant, the same constant obtained when P is above L. This shows that the trajectory satisfies Theorem 3, so it is a parabola with focus at F.

Modified pursuit problem The problem can be modified so that the parabola is replaced by other conics. Specifically, suppose a wind of constant speed v blows radially outward from a given point F_0 different from F. This particular application may not conform to reality, but other more realistic physical situations can be imagined that involve the same ideas. In this case one can verify, with analysis similar to that given above, that the aircraft moves along a portion of an ellipse. If the wind blows radially inward toward F_0, then the aircraft moves along a portion of a hyperbola. In both cases the foci are at F_0 and F.

Concluding remarks

Replacing a focus of a conic by a focal circle is a very simple idea that has profound consequences. It allows us to obtain new characteristic properties of central conics and to extend them to a parabola, and *vice versa*.

The classical characterization of an ellipse as the locus of points whose sum of focal distances $f_1 + f_2$ is constant, and the hyperbola as the locus of points whose absolute distance $|f_1 - f_2|$ is constant, has been generalized to a common bifocal property $|d_1 \pm d_2|$ is constant, where d_1 and d_2 are the shortest distances from a point to the focal circles. By allowing the radius of one of the focal circles to become infinite, we obtained corresponding properties for the parabola. We also introduced special pairs of focal circles, called circular directrices, which provide equidistant properties for central conics analogous to the classical focus-directrix equidistant property for the parabola.

It should be mentioned that in an earlier paper [**1**] we replaced the foci of a central conic by circular *disks*, called focal disks, whose centers are not necessarily at the foci, and found a new set of properties that characterize the conics in terms of the sums and differences of tangent lengths to the focal disks. This characterization occurs naturally when the conics are regarded as sections of a twisted cylinder, of which the circular cone is a special case.

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Summary We introduce a string mechanism that traces both elliptic and hyperbolic arcs having the same foci. This suggests replacing each focus by a focal circle centered at that focus, a simple step that leads to new characteristic properties of central conics that also extend to the parabola.

The classical description of an ellipse and hyperbola as the locus of a point whose sum or absolute difference of focal distances is constant, is generalized to a common bifocal property, in which the sum or absolute difference of the distances to the focal circles is constant. Surprisingly, each of the sum or difference can be constant on both the ellipse and hyperbola. When the radius of one focal circle is infinite, the bifocal property becomes a new property of the parabola.

We also introduce special focal circles, called circular directrices, which provide equidistance properties for central conics analogous to the classical focus-directrix property of the parabola. Those familiar with paper-folding activities for constructing an ellipse or hyperbola using a circle as a guide, will be pleased to learn that the guiding circle is, in fact, a circular directrix.

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